GCSE Maths Notes

A Haynes MSc

GCSE MATHS NOTES

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INTRODUCTION - Table Of Contents

My GCSE Maths Notes:

- Are self-contained i.e. do not assume prior knowledge of the subject (they are not simply brief revision notes)
- Are somewhat longer than the notes I've provided students with in class, but (deliberately) not as long as typical equivalent textbooks
  - They are about 200 sides of A4 if printed out in the webpage format that I wrote them
- Cover the subject content of a UK syllabus I've taught a number of times
  - Your syllabus may be different, but a lot of the material will probably still be relevant

The contents of the GCSE Maths Notes is summarised below:
(For clickable links, view Contents):

- **Part 1: NUMBER** - Numbers, Money - Chapters 1 - 6
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- **Part 3: SHAPE, SPACE AND MEASURES -** Chapters 13 -20
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Note - if you're not familiar with UK qualifications - GCSEs are typically aimed at 14-16 year olds

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Note - Some ebook (kindles etc.) readers (including phones) *auto-shrink* some items on screen:

- If something looks *very small* on your device (an *image* , an *equation* or *text*) - you may be able to:
  - Enlarge it by (double) clicking it - and (double) click it again to shrink it

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GCSE MATHS NOTES

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GCSE Maths Notes

Part 1: NUMBER - contents

Numbers (Chapters 1 to 4)

Chapter 1

- TYPES OF NUMBERS
- DIRECTED NUMBERS
- THE DECIMAL SYSTEM

- ***Please Note*** - If an *image* , an *equation* or *text* appears *very small* on your device - you may be able to:
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TYPES OF NUMBERS - contents

Numbers fall into certain groups, some of which are given names:

- *Counting numbers* : 1, 2, 3, 4, ...
- *Natural numbers* : 0, 1, 2, 3, 4, ...
- *Integers* : ...., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...

The above series of numbers all continue indefinitely.

- *Rational numbers* are those which can be expressed as fractions, i.e. one number divided by another

Since dividing any number by 1 leaves the number unchanged, all the above numbers are rational numbers because we could express each one as, for example:

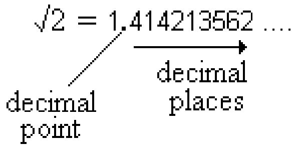
- 1/1, 2/1, 3/1 etc.

The following are also rational numbers:

- proper fractions: Eg. $\frac{1}{2}$, $\frac{2}{3}$ etc. (the numerator (on top) is less than the denominator)
- improper fractions: Eg. $\frac{3}{2}$, $\frac{5}{3}$ etc. (the numerator is more than the denominator)
- mixed numbers: Eg. $1\frac{1}{2}$, $4\frac{1}{5}$
- terminating decimals: Eg. 1/2 = 0.5 exactly
- recurring decimals: Eg. 1/3 = 0.3333333333...... (we write this as $0.\dot{3}$) dot on top

  1/9 = 0.1111111111........... (= $0.\dot{1}$)

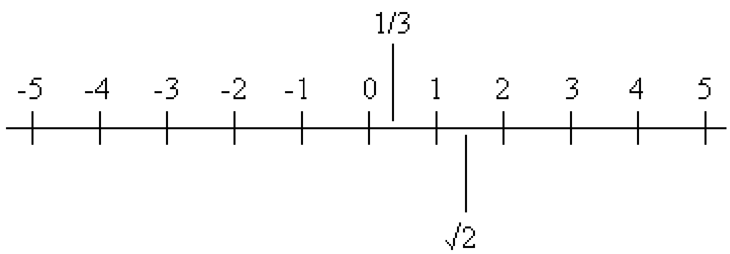
Some numbers cannot be expressed as exact fractions. For example, we say that the square root of 9 is 3, because 3x 3=9. But what about the square root of, for example, 2? On the calculator you will get something like:



The decimal places carry on forever without repeating, which means that the number cannot be expressed as a fraction, i.e. it is not a rational number. Hence, we describe the square root of 2 is an *irrational number* .

- Rational and irrational numbers together form what are called *real numbers*

Any real number, rational or irrational, can be located on the *real number line* - this is represented below - it extends forever to right and left:



Changing a recurring decimal into a fraction

This is illustrated with a couple of examples:

Example

Change $0.\dot{7}$ to a decimal.

We make use of the fact that $\frac{1}{9} = 0.111111\ldots = 0.\dot{1}$

$$0.\dot{7} = 0.777777\ldots = 7 * 0.111111\ldots = 7 * 0.\dot{1}$$

$$= 7 * \frac{1}{9}$$

$$= \frac{7}{9}$$

in these notes we will often use a star (*) to stand for 'multiply' or 'times'
- by hand we would write a cross (x)

Example

Change $0.\dot{4}\dot{7}$ to a decimal.

We make use of the fact that $\frac{1}{99} = 0.01010101.... = 0.\dot{0}\dot{1}$

$$0.\dot{4}\dot{7} = 0.47474747..... = 47 * 0.01010101.... = 47 * 0.\dot{0}\dot{1}$$
$$= 47 * \frac{1}{99}$$
$$= \frac{47}{99}$$

Arithmetic

Arithmetic is the branch of maths concerned with numerical calculations.

There are *four basic operations with numbers* :

- *Addition* , which produces the *sum* of two numbers:

The sum of 2 and 5 equals 7, which we write as: 2 + 5 = 7

- *Subtraction* , which produces the *difference* between two numbers:

The difference between 9 and 5 is 4, which we write as: 9 - 5 = 4

- *Multiplication* , which produces the *product* of two numbers:

The product of 3 and 4 is 12, which we write as: 3 * 4 = 12

[Again note that we here use the star (*) to stand for 'times' or 'multiply by' - whereas by hand we would usually write a cross (**x**)]

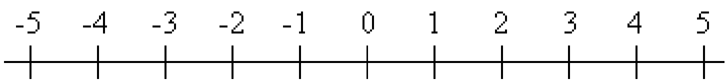
- *Division* , which produces the *quotient* of two numbers:

The quotient of 8 divided by 2 is 4, which we write as: $8 \div 2 = 4$ (we also write $8 \div 2$ as 8/2)

[Notice that the quotient $8 \div 2$ is <u>not</u> the same as $2 \div 8$]

DIRECTED NUMBERS - start of this chapter - contents

Directed numbers are positive or negative numbers. When you add, subtract, multiply or divide directed numbers, you are moving from one point to another along the number line:



The *four basic operations* with directed numbers can be considered in pairs:

1. Addition and subtraction

- 1 + 3 = 4. We can think of this as an instruction to start at number +1 on the number line, then move 3 in the positive direction, bringing us to number +4
- 2 - 5 = -3. We can think of this as an instruction to start at number +2, then move 5 in the negative direction, bringing us to number -3
- -2 + 5 = 3. We can think of this as an instruction to start at number -2, then move 5 in the positive direction, bringing us to number +3

Notice that:

- If there is no sign put in front of a number then it is positive. E.g. 1 and 2 mean +1 and +2

- It may help to change the order of the numbers. E.g. -2 + 5 is the same as +5 -2 = 3

What would, for example, 3 - (-4) mean? The rule here is that:

- *When two minus signs occur together, change them into a plus* , so: 3 - (-4) = 3 + 4 = 7

Notice the use of the $\pm$ button on the calculator in working out $3-(-4)$: The sequence is:

- clear the calculator (turning it off and on should do this)
- enter 3
- press −
- enter 4
- press $\pm$ (this changes +4 to −4)
- press =

You should get the answer 7.

Other examples:

- -5 - 2 = -7
- -2 + 7 = 7 - 2 = 5
- 3 + 5 - 6 = 8 - 6 = 2
- 5 - (-4) = 5 + 4 = 9

With practice, it is not necessary to think in terms of the number line in order to work out the answers.

Example

What is the increase in temperature if it rises from -10^{o} F (F=Fahrenheit) to 65^{o} F?

- increase = final value - initial value = 65 - (-10) = 65 + 10 = 75^{o} F

2. Multiplication and division

- Division may be represented as, for example: $\frac{6}{3}$ or 6/3 or $6 \div 3$

When we multiply or divide two numbers, we have to take into account their signs. When two numbers are multiplied or divided:

- the answer is positive if both number have the same sign (both + or both -)
- the answer is negative if they have opposite signs (one is + and the other is -)

Again, remember that if a number is not written with a sign, then it is taken as being positive:

- 3 * 3 = (+3) * (+3) = + 9
- (-3) * (-3) = + 9
- 3 * (-3) = (+3) * (-3) = -9
- (-3) * 3 = (-3) * (+3) = -9
- 6/(-2) = - 3
- (-6)/2 = -3
- (-6)/(-3) = +2

So,

- a plus and a plus give a plus
- a minus and a minus give a plus
- a plus and a minus give a minus

If there are several numbers to multiply or divide, you can deal with them two at a time. For example:

- 3 * (-4) * (-5) = (-12) * (-5) = + 60

On the calculator the sequence is:

- clear the calculator (turning it off and on should do this)
- enter 3
- press x (multiply button)
- enter 4
- press +/- (this changes +4 to −4)
- press x
- enter 5
- press +/- (this changes +5 to −5)
- press =

You should get the answer 60.

Order of operations

If you enter the following on your calculator, you should get the result shown:

- 10 - 2*3 = 10 - 6 = 4

On the calculator, the sequence is:

- clear the calculator
- enter 10
- press -
- enter 2
- press x (the multiply button)
- enter 3
- press =

You should get the answer 4.

Try the following on a calculator (for the stroke /, press the divide button):

- 3*4/2 - 9/3 = 6 - 3 = 3

On the calculator the sequence is:

- enter 3
- press x
- enter 4
- press ÷ (the divide button)
- enter 2
- press −
- enter 9
- press ÷
- enter 3
- press =

You should get the answer 3.

The calculator performs multiplication and division before addition and subtraction. For example, in the first calculation, it works out 2*3 = 6 and *then* works out 10 - 6 = 4.

Suppose that in the first example, that we really wanted the calculator to work out (10 - 2) and *then* multiply by 3. We would write this as:

- (10 - 2)*3 = 8*3 = 24

On the calculator the sequence is:

- press ([open bracket]
- enter 10
- press -
- enter 2
- press)   [closed bracket]

- press x (the multiply button)
- enter 3
- press =

You should get the answer 24.

The above are examples of applying the BODMAS rule. This stands for:

- Brackets Off
- Division/Multiplication
- Addition/Subtraction

This is intended as a reminder of the *order* numerical operations are performed. The contents of brackets are dealt with first, and divisions and multiplications are done before additions and subtractions.

THE DECIMAL SYSTEM - start of this chapter - contents

The decimal number system is based on the number 10 - presumably it developed because we have 10 digits (fingers and thumbs).

The numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are referred to as *digits* .

When we write a number such as 457 we are using a shorthand notation:

457 = 400 + 50 + 7 = 4 hundreds + 5 tens + 7 units

When we write a number such as 0.637, again we are using a shorthand notation:

- The dot is called the *decimal point* (sometimes, we may use a dash (-), especially on cheques)

- The figures after the decimal point are *decimal places*

The decimal places are fractions with a multiple of 10 on the bottom (the denominator):

$$0.637 = 0.6 + 0.03 + 0.007 = \frac{6}{10} + \frac{3}{100} + \frac{7}{1000}$$

Multiplying and dividing by multiples of 10

An advantage of the decimal system is that it is very easy to multiply or divide by a multiple of 10 - we simply move the decimal point to the right or left.

When multiplying by 10 or 100 or 1000 etc., we move the decimal point to the *right* by the number of zeros:

- 345.678 * 10 = 3456.78 (1 zero in 10, so we move the decimal point 1 place to the right)
- 345.678 * 100 = 34567.8 (2 zeros in 100 so we move the decimal point 2 places to the right)
- 345 * 100 = 34500 (if a decimal point is not shown, it is at the end: 345 *100 = 345.0 *100 = 34500)

When dividing by 10 or 100 or 1000 etc., we move the decimal point to the *left* by the number of zeros:

- 345.6/10 = 34.56 (1 zero in 10, so we move the decimal point 1 place to the left)
- 345.6/100 = 3.456 (2 zeros in 100 so we move the decimal point 2 places to the left)
- 0.34/100 = 0.0034
- 34/1000 = 0.034 (if a decimal point is not shown, it is at the end: 34/1000 = 34.0/1000 = 0.034)

Note that 0.1 = 1/10 , 0.01 = 1/100 etc., so, for example:

- 345.6*0.1 = 345.6/10 = 34.56

Addition

As an example, we work out: 378 + 267

$$\begin{array}{r} 378 \\ 267 \\ \hline 645 \\ \hline \end{array}$$

(1) 7 + 8 = 15 - write the 5, and 'carry' the 1 (keep it in mind, or write it)
(2) 6 + 7 = 13, add the 1 carried, making 14 - write the 4, and carry the 1
(3) 2 + 3 = 5, add the 1 carried, making 6 - write 6

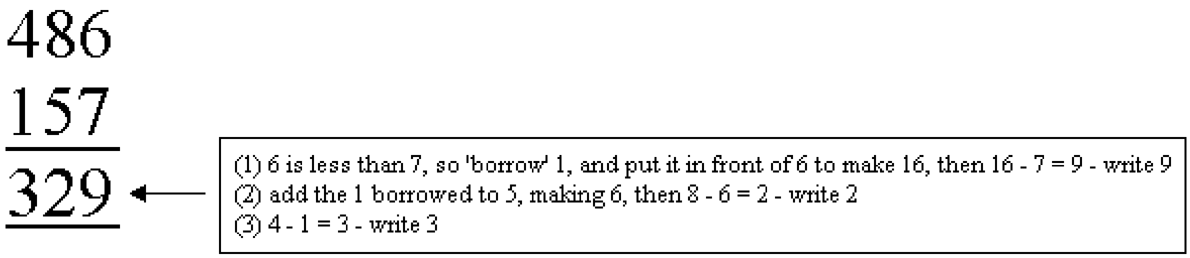
In the next example, the numbers to be added have decimal places. In this case, we put the decimal points directly under each other, and put a decimal point in the answer directly under them - then we add the two numbers in the same way as before.

We work out: 725.85 + 539.4 (notice below that we add an extra zero after the .4 - this is not essential, but it may help if both numbers have the same number of decimal places):

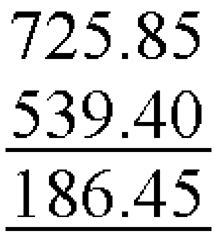
$$\begin{array}{r} 725.85 \\ 539.40 \\ \hline 1265.25 \\ \hline \end{array}$$

Subtraction

As an example, we work out: 486 - 157

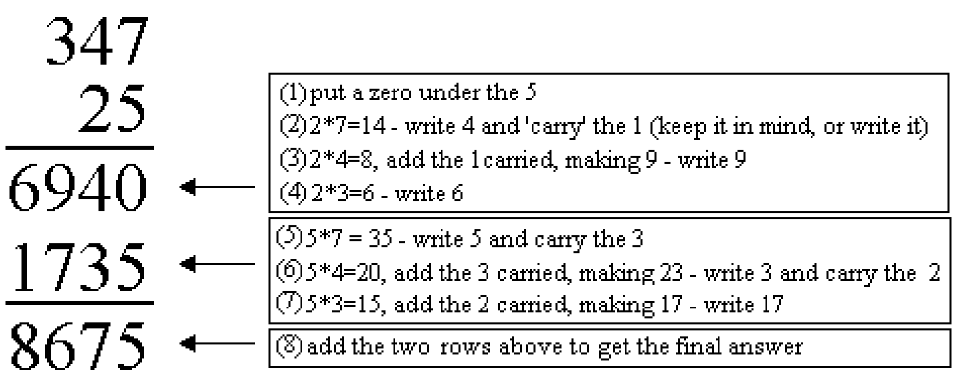


When the two numbers have decimal places, we put the decimal points directly under each other, and put a decimal point in the answer directly under them - then we subtract the two numbers in the same way as before.



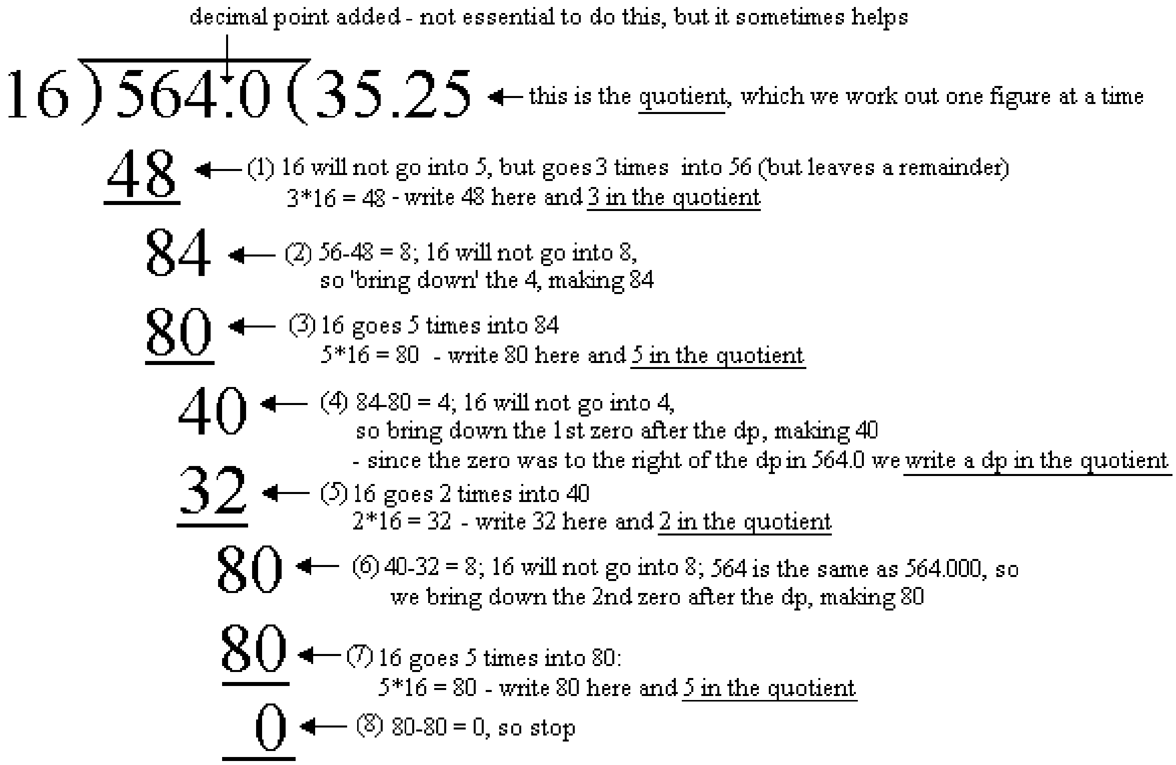
Long Multiplication

As an example, we work out: 347 * 25



Long division

As an example, we work out: 564 ÷ 16 (or we could write: 564/16)



It may be that remainder never reaches zero - in which case the division is continued until a desired number of decimal places have been produced.

Note:

It is usually easier to divide by whole numbers. So if you were dividing, for example, 46.28 by 3.7, first of all multiply both numbers by 10, so you are then dividing 462.8 by the whole number 37.

Part 1: NUMBER - contents

Numbers (Chapters 1 to 4)

Chapter 2

- NUMBER PATTERNS
- FACTORS AND MULTIPLES
- FRACTIONS

NUMBER PATTERNS - contents

Even numbers

All these can be divided exactly by 2:

2, 4, 6, 8, 10, ...

If we represent these as below, we can see that there is a general rule for finding an even number:

place number	1	2	3	4	. . . . . .	n
even numbers	2	4	6	8	. . . . . .	2n

In words, we see that:

- the 1st even number = 2 = 2 * 1
- the 2nd even number = 4 = 2 * 2
- the 3rd even number = 6 = 2 * 3
-
- The nth even number = 2*n which we write as 2n

We can replace the n by any whole number to produce an even number:

- For example, when n = 43, then 2n is the even number 2*43 = 86

Odd numbers

None of these can be divided exactly by 2:

1, 3, 5, 7, 9, 11, ...

If we represent these as below, we can see that there is a general rule for finding an odd number:

place number	1	2	3	4	......	n
odd numbers	1	3	5	7	......	2n -1

In words, we see that:

- the 1st odd number = 1 = (2 * 1) - 1 [the bracket is to shows that the * is done before the -]
- the 2nd odd number = 3 = (2 * 2) - 1
- the 3rd odd number = 5 = (2 * 3) - 1
-
- The nth odd number = (2*n) - 1 which we write as 2n - 1

We can replace the n by any whole number to produce an odd number:

- For example, when n = 43, then 2n -1 is the odd number 2*43 - 1 = 86 - 1 = 85

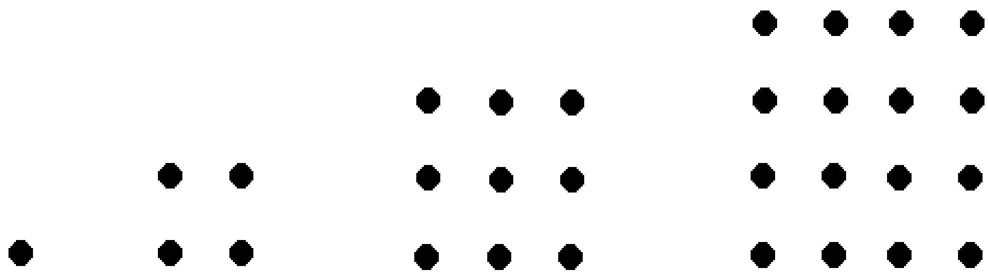
Square numbers

1, 4, 9, 16, ... are square numbers:

Below shows the notation we use for 'one squared', 'two squared', etc..

- 1 = 1 squared = 1*1 = 1^2
- 4 = 2 squared = 2*2 = 2^2
- 9 = 3 squared = 3*3 = 3^2
- 16 = 4 squared = 4*4 = 4^2 etc.
- The nth square number = n^2

We could represent the pattern, or sequence, as:



1 4 9 16

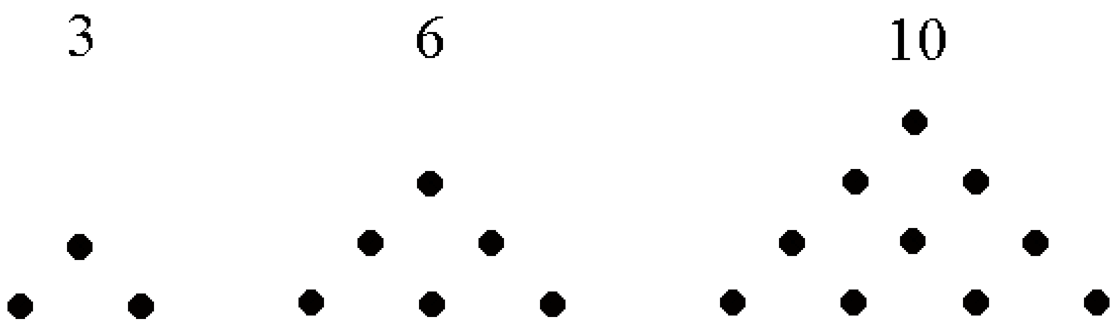
Cube numbers

1, 8, 27, 64,... are cube numbers:

- 1 = 1 cubed = 1*1*1 = 1^3
- 8 = 2 cubed = 2*2*2 = 2^3
- 27 = 3 cubed = 3*3*3 = 3^3
- 64 = 4 cubed = 4*4*4 = 4^3
- The nth cube number = n^3

Triangular numbers

These can be represented by triangles made up of the appropriate number of dots:



Prime numbers

A prime number is any number, greater than 1, that can only be divided exactly by itself and 1:

2, 3, 5, 7, ...

There is not a general rule for working out the nth prime number.

Number sequences are not necessarily whole number sequences, as the next example indicates.

Example

The rule for a number sequence is: $\dfrac{n}{1+n^2}$

What are the first three members of the sequence (i.e. for $n = 1, 2, 3$) ?

When $n = 1$ we get: $\dfrac{n}{1+n^2} = \dfrac{1}{1+1^2} = \dfrac{1}{1+1} = \dfrac{1}{2}$

When $n = 2$ we get: $\dfrac{n}{1+n^2} = \dfrac{2}{1+2^2} = \dfrac{2}{1+4} = \dfrac{2}{5}$

When $n = 3$ we get: $\dfrac{n}{1+n^2} = \dfrac{3}{1+3^2} = \dfrac{3}{1+9} = \dfrac{3}{10}$

FACTORS AND MULTIPLES - start of this chapter - contents

- a *factor* of a number is any integer that divides into the number exactly (i.e. with no remainder)

The factors of 6 are: 1, 2, 3 and 6. (Because 6/1 = 6, 6/2 = 3, 6/3 = 2, 6/6 = 1)

- a *multiple* of a number is that number multiplied by any integer

Some multiples of 6 are: 6, 12, 18, 24, ... (These are 1*6, 2*6, 3*6, 4*6, ...)

Now compare the factors of 6 with the factors of 15, which are: 1, 3, 5 and 15

We see that some factors (1 and 3) the same, i.e. they are *common factors* of both 6 and 15.

- The *highest common factor* ('HCF') of two or more numbers is the highest factor common to the numbers

The HCF of 6 and 15 is 3.

The factors of a number always include 1 and the number itself, but most numbers have other factors as well, as we've seen for 6 and 15. However:

- A *prime number* is a number that can only be divided by 1 and itself - these are: 1, 2, 3, 5, 7, ...

Writing a number as a product of prime factors

Recall that:

- a *factor* of a number is any integer that divides into the number exactly
- a *prime number* is a number that can only be divided by 1 and itself, i.e. 1, 2, 3, 5, 7, ...

Hence:

- a *prime facto* r is a factor which is also a prime number

We can express any number as a product of its prime factors. Suppose we want to express 12 as a product of its prime factors. We do this by dividing by prime numbers as often as possible, starting with 2 - hence:

- dividing 12 by 2 gives 6
- dividing 6 by 2 gives 3 - we cannot divide 3 by 2, so we go to the next prime number, which is 3
- dividing 3 by 3 gives 1 (when we reach 1 we stop)

We've divided by 2 twice and by 3 once, so 12 = 2*2*3 = 2^2 *3

- Note: 2^2 is a shorthand way of writing 2*2. We say 2^2 as '2 to the power 2' - this notation is discussed more later.

Example

Express 90 as a product of its prime factors.

- dividing by 2 gives 45 - this will not divide by 2, so
- dividing 45 by 3 gives 15
- dividing 15 by 3 gives 5 - this will not divide by 3, so
- dividing 5 by 5 gives 1

We've divided by 2 once, by 3 twice and by 5 once, so 90 = 2*3*3*5 = $2*3^2*5$

Lowest common multiple ('LCM')

- The lowest common multiple of two or more numbers is the lowest number that they will both divide into exactly

There are several ways of finding the LCM of two or more numbers:

Method 1 :

Suppose that we want the LCM of 3 and 5. One way is to list the first few multiples of each number:

multiples of 3 are: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, ...

multiples of 5 are: 5, 10, 15, 20, 25, 30, 35, ...

The lists could be continued indefinitely. We see that the first two common multiples of 3 and 5 are 15 and 30.
The lower of these is 15, so 15 is therefore the lowest common multiple (LCM) of 3 and 5.

Method 2 :

Another way to get the LCM of two or more numbers is to:

- multiply the numbers together, which will produces a multiple of all the numbers, then
- see if the multiple can be reduced to a lower multiple by dividing it by 2 or 3 etc.

For example, 2, 3 and 4 multiplied together give $2*3*4 = 24$. So, 24 is *a* common multiple of 2, 3, and 4 since each will divide into 24. Now, dividing 24 by 2 gives us 12, and this can also be divided by 2, 3 and 4. If we divide 12 any more, all three numbers will not divide into it, which means that 12 is the lowest common multiple of 2, 3 and 4.

Method 3 :

We can also find the lowest common multiple of two or more numbers by expressing each number as a product of its primes factors.

Suppose that we want the LCM of 12 and 90.

We already have seen that: $12 = 2^2 *3$ and $90 = 2*3^2 *5$

To get the LCM of 12 and 90 we just multiply the prime factors from either number with the <u>highest</u> power:

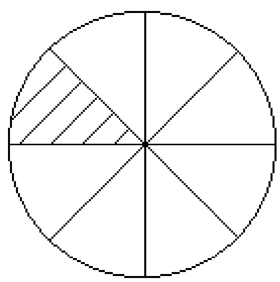
- the highest power of 2 is 2^2
- the highest power of 3 is 3^2
- the highest power of 5 is 5^1 , i.e. simply 5 (5^1 is the same as 5)

The LCM of 12 and 90 is the product of these, so the LCM of 12 and 90 = $2^2 *3^2 *5 = 4*9*5 = 180$.

- To check that 180 *is* a multiple of 12 and 90, we divide it: 180/12 = 15 and 180/90= 2
- Also, 180 is the lowest multiple since, if we try to reduce it by dividing it by 2, say, we get 90 - but 12 does not divide into 90

We can use LCMs when we want to combine fractions - as shown in the next section.

FRACTIONS - start of this chapter - contents



- *A fraction is a part of something*

The above circular pie has been divided into 8 equals parts.

The fraction that is shaded is decribed as 'one eighth' of the whole pie, which we write as $\frac{1}{8}$ or 1/8.

- The bottom number (the *denominator*) says how many equal parts the whole pie is divided into
- The top number (the *numerator*) says how many equal parts of the whole pie are taken

Since the above pie is divided into 8 parts, then:

1/8 means that 1 of the 8 equal parts of the pie is taken
3/8 means that 3 of the 8 equal parts of the pie are taken
8/8 means that all 8 equal parts are taken - the whole pie

A general rule:

- the value of a fraction remains the same if its numerator and denominator are multiplied or divided by the same quantity

For example, multiplying top and bottom of 1/8 by 2 gives 2/16, and these fraction are the same since:

- taking 1 part of a pie divided into 8 equal parts is just the same as taking two parts of a pie divided into 16 equal parts

Also, for example, dividing top and bottom of 6/8 by 2 gives 3/4, and these fraction are the same since:

- taking 6 parts of a pie divided into 8 equal parts is just the same as taking 3 parts of a pie divided into 4 equal parts

Proper and improper fractions

3/8 is and example of a *proper fraction* - the top number is less than the bottom number, and so the fraction has a value less than 1 (3/8 = 0.375).

13/8 is and example of an *improper fraction* - the top number is bigger than the bottom number, and so the fraction has a value greater than 1 (13/8 = 1.625).

An improper fraction can be expressed as a *mixed number*, which is a whole number and a proper fraction.

For example, in the improper fraction 13/8, 8 divides into 13 once, with a remainder of 5, so 13/8 = $1\frac{5}{8}$

Reducing a fraction to its lowest terms

The following fractions are all the same: $\frac{4}{8}$ $\frac{2}{4}$ $\frac{1}{2}$

To change 4/8 to 1/2 we divide both the top and bottom by 2 and then by 2 again.

Simplifying a fraction in this way is called *reducing a fraction to its lowest terms* or *to its simplest form* .

To reduce a fraction to its lowest terms:

- divide the top and bottom by 2 as often as possible
- then divide by 3 as often as possible
- then divide by 5 as often as possible, etc...

Example

Reduce the following fraction to its simplest form: $\frac{60}{144}$

Dividing top and bottom by 2 gives $\frac{30}{72}$

Dividing by 2 again gives $\frac{15}{36}$

2 will not divide, but 3 will, to give $\frac{5}{12}$

No more numbers will exactly divide both top and bottom so $\frac{5}{12}$ is the final answer

As a check , calculate 60/144 and 5/12 on a calculator

Comparing fractions

Which fraction is bigger, **3/5** or **4/7** ?

To compare different fractions, or to put them in order of size, it is easiest if they all have the *same denominator* , since then we only need to look at the size of their numerators.

So, to compare fractions:

- find the LCM of (all) the denominators
- change all the fractions so they all have this LCM as their denominator (it is then called their '*lowest common denominator* ', LCD)

Recall that one way to get the LCM of two or more numbers is to:

- multiply the numbers together, which will produces a multiple of all the numbers, then
- see if the multiple can be reduced to a lower multiple by dividing it by 2 or 3 etc.

So, what about $\frac{3}{5}$ and $\frac{4}{7}$?

We see that 5*7 = 35. This is the LCM of 5 and 7, since we cannot reduce it by dividing by 2 or 3 etc.

Now, 5 divides 7 times into 35, so we times top and bottom of 3/5 by 7:

$$\frac{3}{5} = \frac{3*7}{5*7} = \frac{21}{35}$$

And, 7 divides 5 times into 35, so we times top and bottom of 4/7 by 5:

$$\frac{4}{7} = \frac{4*5}{7*5} = \frac{20}{35}$$

Comparing the numerators, since 21 is bigger than 20, then 21/35 is bigger than 20/35, so 3/5 is bigger than 4/7.

As a check , on your calculator calculate 3/5 and 21/35 to see that they are the same, and calculate 4/7 and 20/35 to see that they are the same.

Combining fractions

A half plus a quarter equals three quarters. In symbols, 1/2 + 1/4 = 3/4.

To produce the answer 3/4, the 1/2 has been changed to 2/4 and then added to the 1/4.

This is an example of a general rule - in order to combine the fractions:

- make their denominators the same (by finding their LCM), and then
- add their numerators

Example

Simplify $\frac{1}{2} + \frac{1}{3}$

We see that 2*3 = 6, which is the LCM of 2 and 3:

- 2 goes 3 times into 6, so multiply top and bottom of 1/2 by 3
- 3 goes 2 times into 6, so multiply top and bottom of 1/3 by 2

$$\frac{1}{2} = \frac{1*3}{2*3} = \frac{3}{6}$$

$$\frac{1}{3} = \frac{1*2}{3*2} = \frac{2}{6}$$

Now the fractions have a common denominator, 6, and we can add their numerators:

$$\frac{1}{2}+\frac{1}{3} = \frac{3}{6}+\frac{2}{6} = \frac{5}{6}$$

At this stage, check to see if the answer can be simplified - by dividing top and bottom by 2 or 3 etc., if possible (in this example it cannot).

As a check , work out (1/2 + 1/3) and then 5/6 on a calculator (you can do this check for the next example too).

Example

Simplify $\frac{1}{3}+\frac{2}{5}$

The LCM of 3 and 5 is 15.

We want 15 to become the common denominator, i.e. the same for both fractions.

To change 3 to 15, we muliply by 5, so we must do the same to the 1 on top, making it 5.

To change 5 to 15, we muliply by 3, so we must do the same to the 2 on top, making it 6.

So, we get:

$$\frac{1}{3}+\frac{2}{5} = \frac{5}{15}+\frac{6}{15} = \frac{5+6}{15} = \frac{11}{15}$$

The process is just the same for subtracting two fractions:

Example

Simplify $\frac{1}{3}-\frac{1}{5}$

The LCM of 3 and 5 is 15, so we get:

$$\frac{1}{3}-\frac{1}{5} = \frac{5}{15}-\frac{3}{15} = \frac{5-3}{15} = \frac{2}{15}$$

Example

Simplify $\frac{1}{12} + \frac{1}{90}$

The LCM of 12 and 90 is 180, so we get:

$$\frac{1}{12} + \frac{1}{90} = \frac{15}{180} + \frac{2}{180} = \frac{15 + 2}{180} = \frac{17}{180}$$

Example

Simplfy $\frac{5}{6} - \frac{4}{5}$

We see that $5*6 = 30$ This is the LCM of 5 and 6.

Now, 6 divides 5 times into 30, so: $\frac{5}{6} = \frac{5*5}{6*5} = \frac{25}{30}$

And, 5 divides 6 times into 30, so: $\frac{4}{5} = \frac{4*6}{5*6} = \frac{24}{30}$

So, $\frac{5}{6} - \frac{4}{5} = \frac{25}{30} - \frac{24}{30} = \frac{1}{30}$

GCSE Maths Notes

Part 1: NUMBER - contents

Numbers (Chapters 1 to 4)

Chapter 3

- PERCENTAGES
- INDICES
- ROOTS

PERCENTAGES - contents

As mentioned before, if we have a pie and divide it into 8 equal parts, then we can have 1/8 or 3/8 etc. of that pie. We say that 1/8 etc. is a *fraction* of the whole pie.

Now, the word '*percent* ' means 'per hundred' or 'out of 100'.

So, when dealing with percentages, we think of the pie (or anything else) as being divided into 100 equals parts, and a stated percentage is a particular number of those parts.

So, for example, 50 percent (written as 50%) means '50 out of 100 equal parts', which is the same as the fraction 50/100 which equals 1/2 which is 0.5

Hence, a percentage is just a special fraction, in which an item is thought of as being divided into 100 equal parts.

Remember that in order to compare different fractions, we had to change them so that they had the same bottom (the same denominator). A useful thing about percentages is that they always have the same bottom, i.e. 100. So we can easily compare different percentages.

We can see at once that, for example, 50% of a pie is bigger than 49% of the pie which is bigger than 25% of the pie etc. Or, for example, a 40% discount off the price of a pair of jeans is better than a 10% discount. Notice that percentages are always percentages of *something in particular* - a pie, the price of a pair of jeans etc.

Using percentages makes calculations easier. For one thing we can easily change a percentage into an ordinary decimal. For example, 45% means 45/100, which is 0.45. So we simply move the decimal point two positions to the left.

So, for example:

75% = 75/100 = 0.75
32% = 32/100 = 0.32
10% = 10/100 = 0.10
4% = 4/100 = 0.04 (*not* 0.4, which is 40%)

Example

What is 60% of £20?

To turn this into the form of a calculation:

- replace 60% with 60/100 = 0.60, and
- replace *of* with *multiply*

So, 60% of 20 = 0.60 * 20 = 12

So, 60% of £20 is £12.

Example

What is 75% of 40 cans of pop?

75% of 40 = 0.75 * 40 = 30

So, 75% 0f 40 cans is 30 cans.

Example

A pair of jeans is priced at £36. They are on special offer with a 60% discount. What is the selling price?

Firstly, what it the discount?

60% of 36 = 0.60 * 36 = 21.6

21.6 is the same as 21.60, so the discount is £21.60

So, the selling price is 36 - 21.60 = 14.4

So, the selling price is £14.40

Calculating a percentage

What is 16 as a percentage of 128? To calculate this, we divide the 16 by 128 and multiply the result by 100.

$$\frac{16}{128} * 100 = 12.5\%$$

So, 16 is 12.5% of 128.

We can check that the answer is correct:

12.5% of 128 = 0.125 * 128 = 16.

Example

A TV is priced at £200. It is decided to reduce it by £30. What is the percentage discount?

$$\frac{30}{200} * 100 = 15\%$$

As a check:

15 % of 200 = 0.15 * 200 = 30

Example

What is 30p as a percentage of £3.

In this sort of calculation both quantities *must have the same units* .

There are 100 pence in £1, so there are 300 pence in £3. We can now calculate the % as:

$$\frac{30}{300} * 100 = 10\%$$

So, 30 pence is 10% of £3.

Increasing by a percentage

Example

A computer is priced at £800, not including 17.5% VAT.

What is the selling price with VAT included?

17.5% of 800 = 0.175 * 800 = 140

So, the selling price = £800 + £140 = £940

Note - when something has to be increased by a certain percentage, a quick way to do it is:

- convert the percentage increase to a decimal
- add this to 1 so you get 'one point something'
- multiply the original number by this 'one point something'

In the above example:

- 17.5% = 17.5/100 = 0.175
- 1 + 0.175 = 1.175
- 800*1.175 = £940

Example

Increase 450 kg by 26%.

- 26% = 0.26
- 1 + 0.26 = 1.26

- so the answer is 450*1.26 = 567 kg

Example

An 80 pence box of cereals increases in price by 5%. What is the new price?

- 5% = 5/100 = 0.05 (*not* 0.5, which is 50%)
- 1+ 0.05 = 1.05
- so the new price is 80 * 1.05 = 84 pence

Decreasing by a percentage

Note:

- *appreciation* is when something's value increases with time - e.g. many art works and houses appreciate
- *depreciation* is when something's value decreases with time - e.g. most cars depreciate

Example

A car is bought for £15000, and a year later has lost 10% of its original value. What is its new value?

- 10% of 15000 = (10/100) * 15000 = 0.10 * 15000 = 1500
- new value = 15000 - 1500 = £13500

Note - as with percentage increases, there is a quick way to do a percentage decrease:

- convert the percentage decrease to a decimal
- take this off 1, to give 'nought point something'
- multiply the original number by this 'nought point something'

In the above example:

- 10% = 10/100 = 0.10
- 1 - 0.10 = 0.90
- New value after a year = 15000*0.90 = £13500

Be careful if the depreciation is less than 10%. Suppose that it had been 5% during the year:

- 5% = 5/100 = 0.05   (*not* 0.5, which is 50%)
- 1 - 0.05 = 0.95
- New value after a year = 15000*0.95 = £14250

INDICES - start of this chapter - contents

- 3*3 is written as 3^2 , and is said as '3 squared' or '3 to the power 2'
- 5*5*5 is written as 5^3 , and is said as '5 cubed' or '5 to the power 3'

In a number expressed like 3^2 :

- the 3 is called the *base* number and
- the 2 is called the *index* of 3 or the *power* of 3 (we say that 3 is raised to the power 2)

The index tell you how many times the base is multiplied by itself.

So, 3^2 tells you that 3 is multiplied by itself 2 times. The value of 3 squared, $3^2 = 3*3 = 9$
So, 5^3 tells you that 5 is multiplied by itself 3 times. The value of 5 cubed, $5^3 = 5*5*5 = 125$

Any number raised to the power 1 is unchanged. So, 3^1 is the same as 3.

Example: What is the value of 2^6 ?

$2^6 = 2*2*2*2*2*2 = 64$

Using a calculator - You may have a calculator with a button marked with y^x (or something similar). This can be used to work out numbers which have powers. The 'y' represents a base number and the 'x' represents the power (or index). In the above case, firstly clear your calculator or turn it off and on, then:

- enter 2 (the base)
- press the y^x button
- enter 6 (the power)
- press equals - you should get the answer 64

Multiplying numbers that have indices

Notice that in the previous example we could bracket the six 2s in different ways:

- 2 * (2*2*2*2*2), which equals $2^1 * 2^5$
- (2*2) * (2*2*2*2), which equals $2^2 * 2^4$
- (2*2*2) * (2*2*2), which equals $2^3 * 2^3$

In all three cases the powers add up to 6, so:

- $2^1 * 2^5 = 2^{(1+5)} = 2^6 = 64$
- $2^2 * 2^4 = 2^{(2+4)} = 2^6 = 64$
- $2^3 * 2^3 = 2^{(3+3)} = 2^6 = 64$

The base is the same, i.e. 2, in each of the above, and we get the general rule that:

- to multiply two numbers with the *same base* we add together their indices (their powers)

Example : What is the value of $2^2 * 2^3 * 2^1$

We could do this as:

$2^2 * 2^3 * 2^1 = 4*8*2 = 64$

Or, we could add the powers to get:

$2^2 * 2^3 * 2^1 = 2^{(2+3+1)} = 2^6 = 64$

Dividing numbers that have indices

Consider:

$$\frac{2^6}{2^2} = \frac{2*2*2*2*2*2}{2*2} = 2*2*2*2 = 16$$

Notice that the answer, $2*2*2*2 = 2^4 = 2^{(6-2)}$

It is a general rule that:

- to divide two numbers with the *same base* we subtract their indices

Example

Evaluate $\frac{2^5 * 2^4}{2^2}$

$$\frac{2^5 * 2^4}{2^2} = \frac{2^9}{2^2} = 2^7 = 2*2*2*2*2*2*2 = 128$$

Example

Evaluate $\frac{2^2 * 2^5 * 3^7}{2^3 * 3^4}$

$$\frac{2^2 * 2^5 * 3^7}{2^3 * 3^4} = \frac{2^2 * 2^5}{2^3} * \frac{3^7}{3^4} = 2^{(2+5-3)} * 3^{(7-4)} = 2^4 * 3^3 = 16 * 27 = 432$$

Notice that:

- we can only combine powers for numbers with the *same base* - so we dealt with the numbers with base 2 separately from those with base 3

Powers raised to powers

Consider: $(2^4)^3$

The 3 outside the bracket means that the quantity inside has to be multiplied by itself 3 times

So, $(2^4)^3 = 2^4 * 2^4 * 2^4 = 2^{(4+4+4)} = 2^{12} = 4096$ (check this on your calculator)

Notice that the final power, 12, equals 4*3, which is the original two powers multiplied together. This is a general rule:

- When a number with a power is raised to a power, multiply the powers together

So, we could have written the calculation as:

So, $(2^4)^3 = 2^{(4*3)} = 2^{12} = 4096$

Now consider: $\left(\frac{4^2}{2^3}\right)^2$

We can apply the rule to top and bottom separately: $\left(\frac{4^2}{2^3}\right)^2 = \frac{4^{(2*2)}}{2^{(3*2)}} = \frac{4^4}{2^6} = \frac{256}{64} = 4$

As a check, we can do the calculation inside the bracket first: $\left(\frac{4^2}{2^3}\right)^2 = \left(\frac{16}{8}\right)^2 = 2^2 = 4$

The power zero

- *Any* number to the power zero equals one

No matter whether it is big, little, positive or negative, if a number is raised to the power 0 then it equals 1.

- $1^0 = 1$
- $2^0 = 1$
- $(-2)^0 = 1$ (some calculators, <u>incorrectly</u> , give an error message when you do this one or the next)
- $(-30)^0 = 1$
- $2000^0 = 1$
- $2279379488^0 = 1$

Reciprocal (or inverse)

- The reciprocal of a number is one divided by that number

So, the reciprocal of $2 = \frac{1}{2} = 0.5$

Since any number to the power zero equals 1, then $2^0 = 1$, so we can write 1/2 as:

- $\frac{1}{2} = \frac{2^0}{2^1} = 2^{0-1} = 2^{-1}$

So, 2^{-1} , '2 to the power minus 1', stands for the reciprocal of 2, i.e.1/2.

Similarly, $3^{-1} = 1/3$

Notice also that, for example:

$$\frac{1}{4^3} = \frac{1^3}{4^3} = \left(\frac{1}{4}\right)^3 = (4^{-1})^3 = 4^{-3}$$

Thus, a minus power, means one over a positive power.

ROOTS - start of this chapter - contents

Square roots

$2^2 = 2*2 = 4$. We say that 4 = 2 squared. We also say that 2 is the *square root* of 4.

Also, since a minus times a minus gives a plus, (-2)*(-2) = 4. So, -2 is also the square root of 4. So,

- the square root of 4 is either +2 or -2

We express the square root of 4 in two ways:

- $\sqrt{4}$
- $4^{1/2}$ (4 to the power 1/2)

You can check these on your calculator - the following should both produce the answer 2:

- Enter 4 and press the symbol $\sqrt{\ }$ (some calculators require you to enter the symbol $\sqrt{\ }$ first)
- Enter 4, then press y^x, then enter 0.5 (which is 1/2) and press equals

A calculator will usually give the positive square root, i.e. 2. The other root is the negative of this, i.e. -2.

Example

What are the square roots of: 25, 81, 16?

$\sqrt{25} = 5$

$\sqrt{81} = 9$

$\sqrt{16} = 4$

Example

Evaluate $\sqrt{9 + 16}$

First work out what is under the root sign: $\sqrt{9 + 16} = \sqrt{25}$. Then, $\underline{\sqrt{25} = 5}$

- Notice that $\sqrt{9 + 16}$ is not equal to $\sqrt{9} + \sqrt{16}$

Cube roots

$2^3 = 2*2*2 = 8$. We say that 8 = 2 cubed. We also say that 2 is the *cube root* of 8.

This time, the negative value is *not* a cube root, since (-2)*(-2)*(-2)= -8, *not* +8.

Again, we can write the cube root of 8 in two ways:

- $\sqrt[3]{8}$ (The 3 is to indicate a cube root is required)
- $8^{1/3}$ (8 to the power 1/3)

Again, the y^x button on a calculator can be used to check that $8^{1/3}$ equals 2. To get the exact answer, you need to be careful how you enter the 1/3, since remember that it is 0.333333......One way is to put the 1/3 inside brackets - try the following:

- enter 8
- press y^x
- press open bracket (
- enter 1/3, i.e. $1 \div 3$
- press close bracket)
- press equals - you should get the answer 2

As a check, when you work out a cube root of a number, cubing the answer should give you the original number.

For example, we find that $8^{1/3} = 2$. Then, as a check we calculate 2^3 , which equals 8.

Other roots

The fourth root, fifth root etc. are written with symbols similar to the cube root, and calculated in a similar manner.

Example

Write down two expressions for the fifth root of 243, and determine its value.

The fifth root of 243 can be written as:

- $\sqrt[5]{243}$, or
- $243^{1/5}$

Answer: $243^{1/5} = 3$

We can check the answer as described earlier, i.e. by working out 3^5 . This equals 243, as expected.

Combining numbers which have fractional indices

The rules for whole number indices also apply when the powers are fractions:

Example

Evaluate $\dfrac{2^{3/4}}{2^{1/4}}$

$$\frac{2^{3/4}}{2^{1/4}} = 2^{(3/4 - 1/4)} = 2^{2/4} = 2^{1/2} = 1.414$$

Sometimes we have letters standing for various quantities, but the rules for indices still apply:

Example

Simplify $\dfrac{a^5\ a^2\ b^{3/4}\ b^{1/4}}{a^3\ b^{1/2}}$ and evaluate it for $a = 2$ and $b = 4$

$$\frac{a^5\ a^2\ b^{3/4}\ b^{1/4}}{a^3\ b^{1/2}} = a^{(5+2-3)}\ b^{(3/4+1/4-1/2)} = a^4\ b^{1/2}$$

$$a^4\ b^{1/2} = 2^4 * 4^{1/2} = 16 * 2 = 32$$

Surds

We can write the following to as many decimal places as we wish, but we cannot write them as exact decimals and we cannot write them as fractions:

$$\sqrt{2} = 1.414 \text{ correct to 3d.p.}$$
$$\sqrt{3} = 1.732 \text{ correct to 3d.p.}$$
$$\sqrt{5} = 2.236 \text{ correct to 3d.p.}$$

Numbers which cannot be written as fractions are called irrational numbers, so the square roots of 2, 3 and 5 (and others) are irrational numbers.

Numbers written in the form $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are called surds.

1. Simplifying expressions containing surds

Example

Express the following in terms of the simplest possible surds:

(a) $\sqrt{32}$

(b) $(2 + 3\sqrt{2})(2 - 2\sqrt{2})$

(a)
$$\begin{aligned}\sqrt{32} &= \sqrt{(16 * 2)} \\ &= \sqrt{16} * \sqrt{2} \\ &= 4 * \sqrt{2} \\ &= 4 * \sqrt{2} \\ &= \underline{4\sqrt{2}}\end{aligned}$$

(b)
$$\begin{aligned}(2 + 3\sqrt{2})(2 - 2\sqrt{2}) &= 4 - 4\sqrt{2} + 6\sqrt{2} - 6(\sqrt{2})^2 \\ &= 4 + 2\sqrt{2} - 6 * 2 \\ &= 4 + 2\sqrt{2} - 12 \\ &= \underline{-8 + 2\sqrt{2}}\end{aligned}$$

Note - as a check, you can work out the values of the original and final expressions on a calculator, to ensure that they are the same.

2. Rationalising a denominator

The surds $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.
However, $(\sqrt{2})^2 = 2$ and $(\sqrt{3})^2 = 3$, so by squaring the surds we change them into rational numbers.

Now, if surds occur on the bottom (the denominator) of a fraction, we can sometimes modify the fraction to change the surds in the denominator into rational numbers, a process called *rationalising the denominator* - as in the following example:

Example

Rationalise the denominator of: $\frac{2}{\sqrt{3}}$

We decide what needs to be done to the denominator to turn it into a rational number, and then do this to top *and* bottom - so in this case:.

We multiply top and bottom by $\sqrt{3}$:

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{2 * \sqrt{3}}{(\sqrt{3})^2}$$
$$= \frac{2 * \sqrt{3}}{3}$$
$$= \frac{2\sqrt{3}}{3}$$

Part 1: NUMBER - contents

Numbers (Chapters 1 to 4)

Chapter 4

- STANDARD FORM
- APPROXIMATIONS
- PROPORTIONAL DIVISION
- DIRECT AND INVERSE PROPORTION

STANDARD FORM - contents

Expressing a number in standard form means writing it as:

- a number between 1 and 10, multiplied by a power of ten

Standard form (also called 'scientific notation') is especially useful when dealing with very large or very small numbers.

- Consider a large number, like 678 000 000 000

The decimal point is not usually shown in a whole number, but it is actually at the end, so:

678 000 000 000 is the same as 678 000 000 000.0

In this case the required number between 1 and 10 is 6.78, and to get this number we have to move the decimal point by 11 positions to the left, and so 11 is the required power of 10.

So, 678 000 000 000 = $6.78 * 10^{11}$ in standard form.

- Now consider a small number, like: 0.000 000 17

In this case the required number between 1 and 10 is 1.7, and to get this number we have to move the decimal point by 7 places to the right. We write this as a *negative* power of 10. So,

- 0.000 000 17 = $1.7 * 10^{-7}$

Example

The speed of light is 300 000 000 metres per second (m/s). Express this in standard form.

300 000 000 is the same as 30 000 000.0

The required number between 1 and 10 is 3.0, and to get this we move the decimal point 8 places to the left. So,

300 000 000 m/s = $3.0 * 10^8$ m/s in standard form.

Example

Multiply $2*10^8$ by $4*10^3$

The rules for multiplying numbers with indices apply to the powers of 10:

- $2*10^8 *4*10^3 = 2*4*10^8 *10^3 = 8*10^{8+3} = 8*10^{11}$

Using a calculator, when you enter $2*10^8$, you do not need to actually enter the 10:

- enter 2
- press EXP

- enter 8

The display on a calculator may depend on its 'mode' setting.

If it is not set to display 'standard form', the display may look like:

200000000

If it *is* set to display standard form, then the display may look like:

2.0 08	Or,	2.0 10^{08}

Both these stand for $2*10^8$.

After the previous step, i.e. entering the 8 on your calculator, now:

- press (x) (the multiply button)
- enter 4
- press EXP
- enter 3
- press equals

You should get the answer:

8.0 11	Or,	8.0 10^{11}

You should always put the base number 10 in a <u>written answer</u> , even if it is not displayed on your calculator:

- Both the above represent: $8*10^{11}$ ('8 times 10 to the power 11').

Example

Evaluate $2.7*10^8 * 3.2*10^{-3}$

On the calculator:

- enter 2.7
- press EXP
- enter 8
- press x (multiply)
- enter 3.2
- press EXP
- enter 3
- press +/- (this changes the power 3 to -3)
- press equals

You should get: $8.64*10^5$ or 864000

Note: Some numbers that occur in science are extremely large or extremely small. For example:

- the mass of the Earth = $6.0*10^{24}$ kg
- the mass of a proton = $1.7*10^{-27}$ kg

It would be very awkward to deal with numbers like these without using standard form notation.

APPROXIMATIONS - start of this chapter - contents

Decimal places (d.p.s)

If you work out the square root of 2 and 5 on a calculator, you will get something like:

The square root of 2, $\sqrt{2} = 1.41421356 \ldots$

The square root of 5, $\sqrt{5} = 2.23606797 \ldots$

Usually, we do not need to write a number with lots of decimal places. Rather we *round* a number (or 'correct' it) to 1 or 2 or 3 decimal places.

If, for example, we only want *two* decimal places, then we look at the *next* decimal place along, i.e. the third decimal place:

- if the third decimal place is less than 5, we leave the second decimal place as it is
- if the third decimal place is 5 or more then we increase the second decimal place by 1

So, to get $\sqrt{2}$ to 2 decimal places, we look at the third decimal place:

$$\sqrt{2} = 1.41421356 \ldots$$

second decimal place   third decimal place

The 3rd decimal place is less than 5, so we leave the 2nd decimal place as it is, so:

$\sqrt{2} = 1.41$ [to 2 decimal places (to 2 d.p., for short)]

To get $\sqrt{5}$ to 2 decimal places, we again look at the third decimal place:

$$\sqrt{5} = 2.23606797 \ldots$$

The 3rd d.p. is 6, so we increase the 2nd d.p. by 1, so:

$\sqrt{5} = 2.24$ (to 2 d.p.)

Note the following - suppose that we want to express 3.6497 to 3 decimal places:

We look at the 4th d.p., which is 7. So we increase the 3rd d.p. by 1. But 9 +1 is 10. In this situation:

- we change the 9 to a 0, and increase the previous number by 1, i.e. change the 4 to a 5

So, 3.6497 = 3.650 (to 3 d.p.)

Significant figures (s.f.s)

Consider the number 482.62

We could write this as:
482.62000000000000000000000000000.........................

However, the extra zeros do not tell us anything, so we say that they are not significant.

We say that 482.62 is stated to 5 significant figures.

If we only want *four* significant figure, we look at the *fifth* figure:

- if the fifth figure is less than 5, leave the 4th figure as it is
- if the fifth figure is 5 or more increase the 4th figure by 1

So, since the fifth figure (the 2) is less than 5, then to 4 significant figures, the number equals 482.6

We say that we have 'rounded' or 'corrected' the number to 4 significant figures, and we may write it as: 482.6 to 4 s.f.

In the same way, the number equals:

- 483 to 3 s.f.s (we've increased the 2 to a 3, since the next figure was a 6)
- 480 to 2 s.f.s
- 500 to 1 s.f.s

Approximate answers

Does 9.875 * 5.123 = 505.9?

We can see that this cannot be correct, because the answer must be about 10*5 = 50.

It is sometimes useful to be able to make an estimate of an answer. In the above case, perhaps the first figure was entered in the calculator as 98.75 by mistake - the approximate answer reveals that a mistake has been made.

When estimating an answer, we may round figures to the nearest 1 or 10 or 100 etc. Sometimes we approximate figures to a certain number of decimal places or to a certain number of significant figures.

Example

Estimate 42*97/9

Rounding each figure to the nearest ten gives us 40*100/10 = 4000/10 = 400

Example

Find an approximate value of 427*52 by correcting the numbers to one significant figure.

Correcting each number to one significant figure, 427*52 is about 400*50 = 20000

Accuracy of measurements

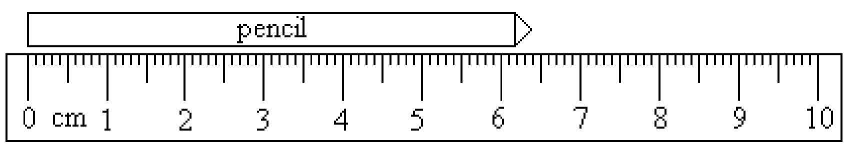
When dealing with money, prices are often rounded up or down to a whole pence, since there is no coin smaller than a whole pence. However, the rounding is not always to the *nearest* whole pence, since this may cost someone money.

For example, a grocer sells bags containing 16 apples for £1-50 (we often use a dash for the decimal point when talking about money). How much would he probably charge for one apple?

- Each apple works out at £1-50/16 = 150p/16 = 9.375pence. If he charged 9pence he would be making a loss (someone could buy 16 apples at 9p and only pay £1-44 in total) - so he would probably round the price up to 10pence each apple (he would then get £1-60 for 16 individual apples).

Rounding numbers up or down is commonly done when measurements are taken:

The pencil in the diagram is 6.4cm long, to the nearest 0.1cm.



A measured length stated as 6.4cm means that:

- if the length appeared to be 6.35cm or above it was rounded up to 6.4cm, or
- if it appeared to be less than 6.45cm its was rounded down to 6.4cm

Whenever a measurement is quoted, the last figure has usually been rounded up or down to the nearest whole number. Thus, a length stated as 6.4 cm actually means that the length is somewhere between 6.35cm and 6.45cm.

Suppose now that a metre ruler was used to measure the sides of a piece of paper, in order to work out its area, and it was found that:

- length = 12.4 cm (so the length was between 12.35cm and 12.45cm)

- width = 8.7 cm (so the width was between 8.65cm and 8.75cm)

Ordinarily we would simply work out the area as = 12.4 * 8.7 = 107.9 cm^2 to 1d.p.

However, what if we take in to account the uncertainty in the measured values?

- If we use the smallest values, we get the area = 12.35*8.65 = 106.8cm^2
- If we use the largest values, we get the area = 12.45*8.75 = 108.9cm^2

So we see that the uncertainly in the individual measurements produces an uncertainly in the result - this is an important consideration for engineers, for example.

PROPORTIONAL DIVISION - start of this chapter - contents

- Proportional division means sharing or dividing something(s) in a given *ratio*

Suppose that we are to divide 20 cans of pop in the ratio 2 to 3 (written as 2:3) between John and Sue.

Adding the numbers in the ratio tells us how many equals parts to divide the 20 cans into:

- 2 + 3 = 5, so there are 5 equal parts, and each part equals 20/5 = 4 cans

John gets 2 sets of these, so he gets 2*4 = 8 cans.

Sue gets 3 sets of these, so she gets 3*4 = 12 cans.

As a check, 8 cans + 12 cans = 20 cans, the original number.

Producing a ratio

We can do the reverse of the above, i.e. if told how many cans each person has, we can work out the ratio.

So, the question would be, if John had 8 cans of pop and Sue has 12 cans, what is the ratio of John's cans to Sue's cans?

The ratio can immediately be written as 8:12

To simplify this as much as possible, we divide *both* numbers until we cannot divide any more, as below:

- Dividing both by 2 gives the ratio 4:6
- Dividing both again by 2 gives the ratio 2:3

We cannot divide any further, so the ratio of John's cans to Sue's is 2:3 (the same as above).

DIRECT AND INVERSE PROPORTION - start of this chapter - contents

Direct proportion

If one bottle of milk is 30p, then two bottles are 60p and 3 bottles are 90p etc.

- When quantities increase at the same rate we say that they are in direct proportion, or that they are directly proportional to each other

The number of bottles of milk and the total price paid are in direct proportion:

cost of 2 bottles = 2 * cost of one bottle
cost of 3 bottles = 3 * cost of one bottle etc....

In general, if one quantity y is in direct proportion to another quantity x, we express this in symbols as:

$$\boxed{y \ \alpha \ x}$$

Greek letter 'alpha' - stands for 'is directly proportional to'

We can replace the proportional symbol by an equals sign if we add a 'constant of proportionality', which we will here call k:

$$\boxed{y = k\,x}$$

We can see from this that:

$$\boxed{k = \frac{y}{x}}$$

We can use any corresponding values of y and x to calculate k using k = y/x. Then we can use y = kx to find y for any x value.

Example

Suppose that 5 apples cost 75p. How much would 8 apples cost.

Let y = kx, where y = cost of x apples. Given that when y = 75p, then x = 5, then:

- k = y/x = 75/5 = 15p (price per apple)
- y = 15x, so when x = 8, y = 15*8 = 120 pence (=£1-20)

Inverse proportion

Suppose that you walk to town, and it takes you 30 minutes. If you walked twice as fast, then it would take you 15 minutes. So when the speed increases the time taken decreases.

We say that the speed and the time to cover a given distance are in inverse proportion, or that they are inversely proportional to each other.

In general, if one quantity y is in inverse proportion to another quantity x, we express this in symbols as:

$$\boxed{y \propto \frac{1}{x}}$$

As before, we can replace the proportional symbol by an equals sign if we add a constant of proportionality, which we will again call k:

$$y = k * \frac{1}{x}$$

Or, $$\boxed{y = \frac{k}{x}}$$

We can see from this that:

$$\boxed{k = y\,x}$$

We can use any corresponding values of y and x to calculate k.

Example

Two ground staff take 3 hours to cut the grass on football pitch. How long would 5 ground staff take for the same job?

If the staff were doubled, we would expect the time taken to be halved, so the number of staff and the time taken are in inverse proportion.

Now, we don't have to use y and x to represent the variables. Suppose in this case that we let:

- N = number of staff, and
- T = time taken for the job

Then, T = k/N. Given that T = 3 hours when N = 2, then:

- k = NT = 2*3 = 6, so
- T = 6/N, so when N = 5, T = 6/5 = 1.2 hours

There are 60 minutes in an hour, so 1.2 hours = 1.2*60 = 72 minutes.

So, 5 ground staff would take 72 minutes to cut the grass.

GCSE Maths Notes

Part 1: NUMBER - contents

Money (Chapters 5 and 6)

Chapter 5

- PAY
- TAX
- HOUSEHOLD EXPENSES

PAY - contents

Hourly pay

1. Peter works in a factory and is paid at £4-00 per hour. How much does he earn for a 40 hour week?

Pay = number of hours * hourly rate = 40 * 4 = £160

Note - £160 is referred to as his 'gross pay' - before any stoppages (tax etc.)

2. Sue gets paid £57.60 for 16 hours in a part-time job. What is her hourly rate of pay?

$$\text{Hourly rate of pay} = \frac{\text{total pay}}{\text{number of hours worked}} = \frac{57\text{-}10}{16} = £3\text{-}60 \text{ per hour}$$

Overtime

Overtime is any time worked beyond a certain number of basic hours, often 40 hours.

Overtime is often paid at:

- 'time and a quarter' - for example, for 4 hours overtime, 5 hours are paid (5 = 4*1.25)
- 'time and a half' - for example, for 4 hours overtime, 6 hours are paid (6 = 4*1.5)
- 'double time' - for example, for 4 hours overtime, 8 hours are paid (8 = 4*2)

Example

Peter gets paid £4-00 for his first 40 hours of work and gets paid time and a half for overtime. One week he does 50 hours of work. How much is his gross pay?

We need to work out the pay for the basic hours and then the extra pay for the overtime:

- basic pay = 40 * 4 = £160
- for 10 overtime, Peter gets paid for 10*1.5 = 15 hours pay = 15 * 4 = £60
- total pay = 160 + 60 = £220

Example

Eleanor's basic hourly rate is £5.20.

One week she does 6 hour 30 minutes overtime which is paid at time and a half.

How much is her overtime pay?

- There are 60 minutes in an hour, so 30 minutes = 20/60 = 0.5 hours
- So her overtime is 6.5 hours
- 6.5 hours at time and a half is 6.5*1.5 = 9.75 hours
- 9.75 hours at £5.20 per hour is 9.75*5.30 = £50.70

So, her overtime pay is £50.70.

Piecework

In some industries employees are paid for each item they produce - this is piecework. Sometimes piecework is paid as a 'bonus', in addition to a basic wage, for production over a specified level.

Example

Mary constructs toy dolls. She gets a basic wage of £130 for a five day week. She gets a 50p bonus for every doll over 80 she makes per day. The following is her daily production for one particular week - what was her gross pay that week?

Dolls Mary made from Monday to Friday: 100, 110, 80, 90, 90

- The total number of dolls made over 80 = 20 + 30 + 0 + 10 + 10 = 70
- Bonus pay = 70 * 50 = 3500 pence = £35   (divide pence by 100 to get pounds)
- Total gross pay = basic wage + bonus = 130 +35 = £165

Commission

Salespeople, for example, are often paid a percentage of the value of the items they sell, sometimes in addition to a basic wage.

Example

Stewart sells double glazing. He does not get a basic wage, but gets 12.5% of the value of the windows he sells. One week he makes sales of the value of £1200, £800, £600, £1400, £1000. What is his commission for that week?

- Total value of sales = 1200 + 800 + 600 + 1400 + 1000 = £5000
- Commission (Stewart's pay) = 12.5% of 5000

Recall that

- 12.5% means = 12.5/100 = 0.125
- and 'of' means 'multiply'

So, Stewart's commission = 0.125 * 5000 = £625

Salaries

People such as teachers and civil servants are paid a certain amount for a year's work, called their annual salary. This is usually divided into 12 equal monthly payments.

Example

A civil servant's gross pay is £1600 per month. What is his gross annual salary?

gross annual salary = pay per month * 12 = 1600 * 12 = £19200

TAX - start of this chapter - contents

The Chancellor of the Exchequer raises money for government spending by taxation - this includes:

- Direct taxation - Income tax, i.e. tax on a person's income
- Indirect taxation - VAT ('Value Added Tax'), which is an amount added to most goods and services that we buy

PAYE ('Pay As You Earn')

This is the form of tax most people pay who are employed. The tax is deducted before they receive their pay.

Before working out the amount of tax to be paid, some *tax allowances* are taken from the gross income to find the *taxable income* .

Note: The tax rules can be changed each time there is a budget - so any figures stated below are not meant to be exact, but are just used for illustration.

Tax allowances include, for example:

- a single person's allowance - E.g. £4500
- a married couple's allowance - E.g. £3000
- allowance for dependants

The married allowance is often added to the husband's personal allowance, but can be transferred to his wife.

If we know a person's total tax allowances, then:

- Taxable income = gross income - tax allowances

Tax on the taxable income is paid at different rates.

- on the first £3000 of taxable income, tax is charged at 20 pence in the pound (i.e. 20%), and is the *lowest rate* of tax
- on taxable income between £3000 and £25000, tax is charged at 25%, and is called the *basic rate* of tax
- on taxable income above £25000, tax is charged at 40%, called the *higher rate* of tax

Once the tax is known, then:

- *net* pay = *gross* pay - tax

Note that there are other deduction made as well as income tax, such as National Insurance.

Example

John is married and has an annual salary of £20500. His wife isn't working. Using the figures quoted above for allowances and tax rates, calculate:

1. his tax allowance
2. his taxable income
3. the total tax he pays in the year
4. his net pay per month, ignoring any other deductions

1. Since John's wife is not working, he will get his personal allowance and the married couple allowance, so:

- tax allowance = 4500 + 3000 = £7500

2. His taxable income is his gross salary less his tax allowance, so:

- taxable income = 20500 - 7500 = £13000

3. John will pay:

- tax at lowest rate, 20%, on the first £3000 of his taxable income
    - tax due = 20% of 3000 = 0.20 * 3000 = £600

- tax at the basic rate, 25%, on the rest of his taxable income, i.e. on £10000 (=13000-3000), so
    - tax due = 25% of 10000 = 0.25 * 10000 = £2500

Total tax for the year = 600 + 2500 = £3100

4. John's net annual pay = gross pay - tax = 20500 - 3100 = 17400

- Net pay per month = 17400/12 = £1450

VAT ('Value Added Tax')

VAT is added to many (but not all) goods and services.

Example

A computer is priced at £1000, excluding VAT.

What is the selling price with VAT at 17.5% added?

- 17.5% of 1000 = 0.175 * 1000 = 175
- So, the selling price = £1000 + £175 = £1175

Recall that a quick way to add a percentage, such as the above is to:

- change 17.5% to a decimal: 17.5% = 17.5/100= 0.175
- add this to 1, which gives 1.175
- multiply 1000 by this, to give 1000*1.175 = £1175

In the above example, notice that:

$$\boxed{\text{price including VAT} = \text{price without VAT} * 1.175}$$

This is useful if we want to reverse the process, since if we divide both sides by 1.175 we get:

$$\frac{\text{price including VAT}}{1.175} = \frac{\text{price without VAT} * \cancel{1.175}}{\cancel{1.175}}$$

$$\frac{\text{price including VAT}}{1.175} = \text{price without VAT}$$

Or, $\boxed{\text{price without VAT} = \dfrac{\text{price including VAT}}{1.175}}$

Applying this to the previous example:

$$\text{price without VAT} = \frac{1175}{1.175} = 1000$$

Remember that the '.175' in '1.175' comes from the VAT rate. If the rate were, for example:

- 12% we would use 1.12 (= 1 + 12/100 = 1 + 0.12)
- 8% we would use 1.08 (= 1 + 8/100 = 1 + 0.08) [*not* 1.8]

HOUSEHOLD EXPENSES - start of this chapter - contents

Comparing prices

Example

Company A sells boxes containing 200 tea bags for £2-50.
Company B sells boxes of 250 tea bags for £3-00. If the only consideration is the price, which of these is better value?

- One way of looking at this is to work out the cost per tea bag:

Company A charges £2-50 (=250p) for 200 bag: cost per tea bag = 250/200 = 1.25 pence
Company B charges £3-00 (=300p) for 250 bag: cost per tea bag = 300/250 = 1.20 pence

Thus, company B's tea bags are a little cheaper.

- Another way of looking at this is to work out how many tea bags are obtained for one pound:

Company A: tea bags per pound = 200/2.50 = 80
Company B: tea bags per pound = 250/3.00 = 83.3

Obviously, you will not get '.3' of a tea bag, but the calculation shows that you do get a few more per pound from company B.

Electricity

Most people pay for their electricity 'quarterly', i.e. they get a bill every three months.

An electricity bill is made up of three parts:

1. a fixed charge (called a 'standing charge') - this is charged even if no electricity is used
2. a charge for each 'unit' of electricity used [the unit is called a kilowatt hour ('kWh')]
3. VAT added to the *total* of parts (1) and (2)

Example (the figures used here are not meant to be exact, but just for the example)

In the previous quarter, Carol used 1200 units of electricity. Her electricity company charges:

- a standing charge of £13, and
- 8.5 pence per unit of electricity used

VAT is charged at 5%.

How much is her electricity bill?

- standing charge = £13
- cost of units = 1200*8.5 pence = 10200 pence = £102.00 (divide pence by 100 to get £)
- total of above = 13 + 102 = £115

To add VAT of 5%, we can:

Either:

- work out 5% of £115 and add it to the bill:
    - (5/100) * 115 = £5.75
    - total electricity bill, including VAT = 115 + 5.75 = £120.75

Or (slightly quicker):

- multiply £115 by 1.05 (= 1 + 5/100 = 1 + 0.05)
    - total electricity bill, including VAT = 115.00 * 1.05 = £120.75

Gas

Gas bills are calculated in a similar manner to electricity bills, except:

- the 'unit' is a quantity of gas, called a 'therm' (105 therms is about 100 cubic feet)
- the standing charge for gas may soon be scrapped (so if no gas is used, there is no charge)

Mortgages

Most people obtain a mortgage to buy their house from a building society or a bank.

The maximum amount they will lend is typically 90% or 95% of the value of the house. The difference has to be paid by the buyer as a 'deposit'.

Example

John and Sue would like to buy a house valued at £55000. Their bank will lend a maximum of 95% of the house value. What is the maximum the bank will lend them, and how much deposit will they need?

- The maximum the bank will lend = 95% of £55000 = (95/100) * 55000 = 0.95 * 55000 = £52250
- The deposit = 55000 - 52250 = £2750

The maximum amount a mortgage lender will lend also depends on the income of the buyer(s). They will typically lend:

- either 3 times the gross annual income of one person
- or 2.5 times the *joint* gross income of a couple

Example

John's gross annual income is £16000. Sue works part-time and earns £8000 a year. Their bank will lend them a maximum of 2.5 times their joint gross income. Can they buy the house in the last example?

- Joint gross income = 16000 + 8000 = 24000
- Maximum loan = 2.5 * 24000 = 60000

This is more than the £52250 they need for the house.

The amount borrowed to buy a house is called the *capital* . The total repaid includes interest, and is much more than the capital. A mortgage is usually paid back monthly over a *term* of 20 or 25 years. There are two types of mortgage repayments:

- A *repayment mortgage* - each month a single payment is made, consisting of:
    - interest charged on the capital for the month, and
    - a part repayment of the capital
- An *endowment mortgage* - each month two payments are made:
    - one is interest charged on the capital for the month
    - the other is a payment on an insurance policy - the policy is designed to 'mature' at the end of the mortgage term, and its value should pay off the capital

Example

John and Sue take out a 25 year repayment mortgage for £52250 to buy the house mentioned previously.

At the moment, their bank charges £9.20 per month for each £1000 owed on a repayment mortgage taken over 25 years.

How much is their monthly repayment?

- monthly repayment = (52250/1000) * 9.2 = 52.25 * 9.2 = £480.70

Insurance

Insurance is paid to avoid financial hardship arising in certain circumstances.

It is a legal requirement to have insurance before a car can be driven on public roads. The *cover* provided depends on the type of policy:

- third party - this is the minimum insurance required by law - it covers injury to others and/or damage caused by your vehicle
- third party fire and theft - in addition to the above, this provides some cover for your vehicle
- comprehensive - in addition to the above this provides complete cover for your car

Most people, either as part of the monthly payment to their mortgage lender, or to a separate company, pay insurance for one or more of:

- their life - if one partner dies, the whole mortgage can be paid off by the insurance company

- their health/work - in the event of prolonged illness or injury or unemployment, the monthly payments may be paid by the insurance company (but often only for 1 year maximum)
- damage to the house (e.g. fire)
- damage to or theft of content

Example (the premium charges below are not exact, but are just for the purposes of the calculation)

Gordon insures his house for £80000 and the contents for £20000.

His insurance company charges annual premiums of:

- 20 p for each £100 buildings cover
- 90 p for each £100 of contents cover

How much is his total annual premium?

- buildings cover costs (80000/100) * 20 = 800 * 20 = 16000 p = £160 (divide pence by 100 to get £)
- contents cover costs (20000/100) * 90 = 200 * 90 = 18000 p = £180
- total annual premium = 160 + 180 = £340

Part 1: NUMBER - contents

Money (Chapters 5 and 6)

Chapter 6

- CREDIT
- SAVINGS - COMPOUND INTEREST
- TRAVEL

CREDIT - contents

Hire purchase

When goods are bought on hire purchase, a deposit is usually paid, and the balance paid over a period of time. The seller usually adds interest to the balance. If an item is bought on hire purchase, the customer does not own the item until all the repayments have been made - if repayments are not made, the item can be repossessed (in effect, the item is on *hire* until it has been completely *purchased*).

Example

A man buys a video. It has a cash price of £300. He pays a 25% deposit. Interest of 16% per annum (=per year) is added to the balance. Repayments are made in equal monthly instalments over twelve months. How much are the instalments?

$$\text{cash price} = £300$$

$$\text{deposit} = 25\% \text{ of } 300 = 0.25 * 300 = £75$$

$$\text{outstanding balance} = \text{cash price} - \text{deposit} = 300 - 75 = £225$$

$$\text{interest} = 16\% \text{ of } 225 = 0.16 * 225 = £36$$

$$\text{total to repay} = \text{outstanding balance} + \text{interest} = 225 + 36 = £261$$

$$\text{monthly instalments} = \frac{262}{12} = £21.75$$

Flat rate and APR

The interest quoted in the last example is called a *flat rate* of interest. The interest of £36 is equivalent to £3 per month. If the whole amount were repaid at the end of the year, the flat rate would be the true annual rate. However, repayments start after only a month, and so the balance is gradually reduced - so the interest paid is a progressively bigger part of the remaining balance.

The true interest rate, called the *annual percentage rate* (APR) is actually about double the flat rate:

$$\boxed{\text{APR} \approx 2 * \text{flat rate interest}}$$

approximately

Thus, the 16% flat rate in the last example is an APR of about 32%.

Companies are legally obliged to quote interest as APR, so customers can compare the cost of credit.

Bank loans

If you borrow money from a bank or building society you will be charged interest, i.e. they will add a charge to the amount you borrow, which has to be paid in addition to the original amount.

The amount borrowed is called the *principal* .

Simple interest is a percentage of principal.

Example

A man borrows £500 at a simple interest rate of 15%. How much interest does he pay?

15% of 500 = 75

So, he pays £75 in interest. His total repayment is £500 +£75 , which is £575.

When simple interest is added to a loan taken over several years, the interest is calculated on the same original principal amount, so the interest per year is the same, so:

- the total interest = the interest for the first year * the number of years

Example

What is the simple interest on £500 borrowed for 4 years at 15%.

15% of 500 = 75

So, the interest per year is £75.

This £75 is charged for each year of the loan, so the total interest is 4*75 = £300

Overdrafts

An overdraft is an arrangement with a bank that allows a customer to draw money *beyond* the amount in the account. When an account is overdrawn, it can be thought of as a negative balance - money needs to be added to it to bring it back to a positive balance.

For example, Bob has £50 in his account and draws £200. He is now overdrawn by £150. He would need to add £150 to the account to bring it back to zero balance.

Overdrafts are often used as short-term arrangements. They can be an expensive way of borrowing money. Most banks charge a fixed amount per month when an overdraft is used, *plus* interest on the amount overdrawn. So Bob would actually have to pay back more than just the £150 he has overdrawn by.

Credit cards

A credit card allows a customer to spend up to a certain limit using the card. Credit cards can be used as readily as cash with many companies. They can also be used where cash cannot - such as for purchases made over the telephone, and now, more and more, over the Internet.

A credit card holder gets a statement each month showing (at least) the:

- transactions made during the previous month
- interest added
- current balance
- payment due

The interest added is typically about 2% per month (about 27% APR) and the monthly payment is typically 5% of the current balance.

SAVINGS - COMPOUND INTEREST - start of this chapter - contents

Compound interest is the type that you would expect to get from a bank on your savings.

Suppose that a bank pays 7% interest on Jan 1st each year on the current balance in savings accounts.

Suppose that you place £500 in a savings account on Jan 1st 2000.

- On Jan 1st 2001, you will get 7% of the £500 added as interest, which is £35, so your balance becomes £535

- On Jan 1st 2002, the bank adds 7% of the new balance to you account. This is 7% of £535, which is £37.45. Thus, you get interest on the previous year's interest, and this is called compound interest

The following represents what happens to your initial £500 over a few years, if you make no withdrawals:

date	interest £	balance £
1-Jan-2000	0.00	500.00
1-Jan-2001	35.00	535.00
1-Jan-2002	37.45	572.45
1-Jan-2003	40.07	612.52

As an alternate method, we could add the 7% by multiplying by 1.07 (= 1 + 7/100 = 1 + 0.07):

Initial balance = £500:

- balance after year 1 = 500 * 1.07 = £535.00
- balance after year 2 = 535.00 * 1.07 = £572.45
- balance after year 3 = 572.45 * 1.07 = £612.52 (rounded to nearest whole pence)

TRAVEL - start of this chapter - contents

Foreign currency

Example

Jim is going on holiday to Spain. His travel agent buys pesetas ('pts') at a rate or 240 pts per £ (pound sterling) and sells pesetas at a rate of 220 pts per

£

1. Jim buys £150 worth of pesetas for his trip - how many pesetas does he get?
2. He returns with 8400pts. How much does he get back in pounds?

1. For £200 = gets 150 * 220 = 33000 pts
2. For 8400 pts he gets 8400/240 = £35

Time

A day is 24 hours long, and the time of day can be indicated in two ways:

On the 12-hour clock , the day is divided in two halves:

- from midnight to midday (noon) - this is am
- from midday to midnight - this is pm

On the 24-hour clock , the day is a single 24 hour interval. The time is indicated by 4 digits.

If the time is 1.00pm or later, 12 hours is added to the 12-hour clock time to get the 24-hour clock time:

12-hour clock	24-hour clock
7.50 am	0750
11.30 am	1130
12.20 pm	1220
4.25 pm	1625
8.30pm	2030

It is useful to understand the 24-hour clock - if only to correctly understand timetables, which are always written using it.

Part 2: ALGEBRA - contents

Algebraic Expressions and Relationships (Chapters 7 to 9)

Chapter 7

- USING SYMBOLS
- SIMPLIFYING ALGEBRAIC EXPRESSIONS
- FACTORISING ALGEBRAIC EXPRESSIONS

USING SYMBOLS - contents

Peter is 5 years older than his sister Sue. If Sue is 8, how old is Peter? As an equation we could say that:

- Peter's age = Sue's age + 5, so
- Peter's age = 8 + 5 = 13 years

In algebra, we use letters (symbols) to stand for numbers. This allows us to change a problem expressed in words into a problem expressed in symbols.

In the above case suppose that:

- we let S stand for Sue's age and P stand for Peter's age, then
- the information that Peter is five years older than Sue can be expressed as: $P = S + 5$
- and being told that Sue is 8, we get Peter's age from $P = 8 + 5 = 13$ years

Now, suppose that the age of Peter and Sue's father is Sue's age plus twice Peter's age - how old is their father?

If we let F stand for the age of their father, then we get the equation:

- $F = S + 2P$, or we could just as well write this as: $F = 2P + S$

Note that:

- 2P and S are called *terms* in the *algebraic expression* 2P + S
- in the expression 2P, the number 2 is called the *coefficient* of P
- 2P stands for 2*P [again remember that * means 'multiply' in these notes - by hand, we would write a cross (x)]

The process of replacing symbols by particular values is called *substitution* - for the above equation we get:

- $F = 2P + S = 2*13 + 8 = 26 + 8 = 34$ years

Notice that we have to put the multiplication symbol back it when a letter is substituted by a number. If we simply put 13 in place of P in the expressions 2P, we would get 213, which is not the same a 2*13.

We can obtain another expression for the right-hand side:

- we have F = 2P + S, but
- we also have P = S + 5, so
- F = 2(S+5) + S

We can *simplify* the expression on the right (which just means making it as simple as possible) by:

- removing the bracket by multiplying every term inside the bracket by the 2 outside:
  - F = 2(S+5) + S = 2*S + 2*5 + S = 2S + 10 + S
- collecting together *like terms* - which means the 2S and S above, so we get: F = 3S +10

Substituting the value of S, i.e. Sue's age into this should give us the same value of F, her father's age, as before:

- F = 3S +10 = 3*8 + 10 = 24 + 10 = 34 (same as before)

Sometimes a problem may be expressed in words and we have to choose what letters to stand for what quantities, as in the above case. Sometimes, we may have an equation in which the meanings of the symbols are not specified.

For example, if y = 2x + 3, what is y when x = 5?

We do not need to know what x or y stand for in order to perform the substitution:

- y = 2x + 3 = 2*5 + 3 = 10 + 3 = 13

SIMPLIFYING ALGEBRAIC EXPRESSIONS - start of this chapter - contents

Suppose that y = 3x + 2x. We can simplify this by writing it as y = 5x. Nothing has changed - we've merely expressed the algebraic expression on the right in a simpler form.

Notice that if we choose a value for x, say x = 2, then

- the original expression   3x + 2x = 3*2 + 2* 2 = 6 + 4 = 10
- the final expression   5x = 5*2 = 10

This same sort of check can be used no matter how many letters occur in an expression that you are simplifying. Just give the letters values and work out the value of original and final expression. If they are not the same, then you've made a mistake somewhere, and you need to check your working. (this is not an absolute check since, sometimes the answers may be the same even though a mistake has been made - but it is till a worthwhile check).

To simplify algebraic expressions, it is necessary to know the rules that can be applied to them.

Rules for addition and subtraction

The rules that apply to numbers also apply to the letters in algebra:

- If there is no sign put in front of a number then it is positive. E.g. 2 is the same as +2 and a is the same as +a
- When two minus signs occur together, change them into a plus. E.g. 3 - (-4) = 3 + 4, and a - (-b) = a + b

We can only add or subtract like terms, i.e. those with the same *algebraic quantity* following their coefficient, and what we actually do is add or subtract the coefficients of the terms.

Examples

$3x + 5x - 2x = (3 + 5 - 2)x = 6x$
$4x - x = 4x - 1x = (4 - 1)x = 3x$
$4w^2 + 5w^2 = 9w^2$
$3y^2 + 3x + 4y^2 - x = 3y^2 + 4y^2 + 3x - x = (3 + 4)y^2 + (3 - 1)x = 7y^2 + 2x$

Rules for multiplication and division

When two numbers or letters are multiplied or divided then:

- the answer is positive if they both have the same sign (both + or both -)
- the answer is negative if they have opposite signs (one is + and the other is -)

Again, remember that if a number or a letter is not written with a sign, then it is taken as being positive:

- $a * b = (+a) * (+b) = +ab = ab$
- $(-a) * (-b) = +ab = ab$
- $a * (-b) = (+a) * (-b) = -ab$
- $(-a) * b = (-a) * (+b) = -ab$
- $a/(-b) = -a/b$
- $(-a)/b = -a/b$
- $(+a)/(+b) = +a/b = a/b$
- $(-a)/(-b) = +a/b = a/b$

If there are several items to multiply or divide, you can deal with them two at a time - for example:

- a * (-b) * (-c) = (-ab) * (-c) = + abc = abc

Examples

$x * y = xy$

$2x * 3y = 6xy$

$2x * 3xy = 6x^2 y$

$(-2x) * (-4y) = +8xy = 8xy$

$\frac{3xy}{y} = 3x$

$\frac{-4x}{-2x} = 2$

$\frac{6y}{-2} = -3y$

Rules for indices

We may write 3*3*3*3*3 as 3^5 .

We say this a '3 to the power 5' or '3 raised to the power 5'. The number 3 is called the *base* , and the number 5 the *index* or the *power* .

Number with the *same base* can be combined in accordance with certain rules:

Example	General
$4^5 = 4*4*4*4*4$	$a^m = a*a*a*.....$ (m times)
$4^2 * 4^3 = (4*4)*(4*4*4) = 4^5 = 4^{(2+3)}$	$a^m * a^n = a^{(m+n)}$
$(4^2)^3 = (4*4)*(4*4)*(4*4) = 4^6 = 4^{(2*3)}$	$(a^m)^n = a^{(m*n)}$
$\frac{4^5}{4^2} = \frac{4*4*4*4*4}{4*4} = 4*4*4 = 4^3 = 4^{(5-2)}$	$\frac{a^m}{a^n} = a^{(m-n)}$
$4^{-2} = \frac{1}{4^2}$	$a^{-m} = \frac{1}{a^m}$
$4^0 = 1$	$a^0 = 1$ (any number to the power zero equals 1)
$4^{1/2} = \sqrt{4} = 2$ $\quad 8^{1/3} = \sqrt[3]{8} = 2$	$a^{1/n} = \sqrt[n]{a}$ (the n^{th} root eg. square root, cube root etc)

Examples

$$\frac{6x^2}{3x} = 2x$$

$$\frac{6x^2y}{2xy^2} = \frac{3x}{y}$$

$$\frac{27x^2y^3z^2}{3xy} = 9xy^2z^2$$

$(2^3x^2)^2 = 2^6x^4 = 64x^4$ (the power outside the bracket applies to both the factors inside)

Removing brackets

When dealing with just *numbers* , we can reduce a bracket to a single number and then multiply or divide by whatever is outside the bracket, for example:

- 3*(4 + 5) = 3*9 = 27

However, in algebra, the terms in a bracket may be different. In this case, when we remove a bracket, we multiply or divide *each* term inside the bracket by what is outside:

- $3x(4x + y) = 3x*4x + 3x*y = 12x^2 + 3xy$

If there is a minus sign outside a bracket, this is the same as -1, so we multiply each term inside the bracket by -1:

- $-(2x + 3y) = -1*(2x + 3y) = -2x - 3y$
- $-(2x - 4z) = -1*(2x - 4z) = -2x + 4z$ [$-1*(-4z) = + 4z$]

If an expression contains more than one bracket, we remove the brackets first and then simplify if possible:

$-2(x-5y) + 5x(1+y) = -2x + 10y + 5x + 5xy = 3x + 10y + 5xy$

Multiplying two brackets

Consider: $(2x + 1)(x - 3)$

When removing these brackets each term in the first bracket must multiply each term in the second bracket:

$$\begin{aligned}(2x + 1)(x - 3) &= 2x(x - 3) + 1(x - 3)\\ &= 2x^2 - 6x + x - 3\\ &= \underline{2x^2 - 5x - 3}\end{aligned}$$

We could do this as below - the lines help us to keep track of what is being multiplied by what:

$$\begin{aligned}(2x + 1)(x - 3) &= 2x^2 - 6x + x - 3\\ &= \underline{2x^2 - 5x - 3}\end{aligned}$$

Check : As a check, put any value of x in the original and the final expressions - suppose that we use x = 1:

- $(2x + 1)(x - 3) = (2*1 +1)(1-3) = 3 * (-2) = -6$
- $2x^2 - 5x - 3 = 2*1^2 - 5*1 -3 = 2 -5 -3 = 2 - 8 = -6$

Both give the same result - as they should for any x value.

FACTORISING ALGEBRAIC EXPRESSIONS - start of this chapter - contents

The factors of a number are those numbers which divide exactly into it. For example:

- the factors of 6 are: 1, 2, 3 and 6
- the factors of 15 are: 1, 3, 5 and 15

If we 'factorise' a number, we express it in terms of (some of) its factors. For example:

- $6 = 2*3$
- $15 = 3*5$

The *highest common factor* (HCF) of two numbers is the highest number which will divide into both of them. For example:

- the HCF of 6 and 15 is 3

Now consider the expression: $4x^3 + 2x^2$

To factorise this we need to find the HCF of both terms. To find the HCF:

- find the highest number which divides each coefficient - this is 2 in the above case
- find the lowest power of each letter that the terms have in common - in the above case, the terms only have x in common, and the lowest power is x^2
- multiply the above together to get the HCF - so, the HCF $=2*x^2 = 2x^2$

We can now divide both terms by $2x^2$, and place $2x^2$ outside a bracket:

- $4x^3 + 2x^2 = 2x^2 \left(\frac{4x^3 + 2x^2}{2x^2}\right) = 2x^2(2x + 1)$

Thus, the expression has been factorised, into the factors $2x^2$ and $(2x + 1)$.

To check that the result is correct, we can 'multiply out' the expression, i.e. remove the brackets:

$2x^2 (2x + 1) = 2x^2 *2x + 2x^2 *1 = 4x^3 + 2x^2$ (the original expression)

Example

Factorise: $12x^3 y + 3x^2 y^2$.

Using the same procedure as before to find the HCF:

- the highest number which divides each coefficient is 3
- the terms have x and y in common, and the lowest power of each are x^2 and y (= y^1)
- so, the HCF of the terms $= 3*x^2 *y = 3x^2 y$

Dividing both terms by $3x^2 y$, and placing $3x^2 y$ outside a bracket, gives:

- $12x^3 y + 3x^2 y^2 = 3x^2 y(4x + y)$

To check that the result is correct, remove the bracket:

- $3x^2 y(4x + y) = 3x^2 y*4x + 3x^2 y*y = 12x^3 y + 3x^2 y^2$ (the original expression)

Example

Factorise: $ax + bx + ay + by$

In this case we can group the terms:

$ax + bx + ay + by = (a + b)x + (a + b)y = (a + b)(x + y)$

Factorising the difference between two perfect squares

Perfect squares are quantities like 2^2 (= 4) and 3^2 (= 9) etc.

Expressions like a^2 and b^2 are also perfect squares.

Now, observe that $(a + b)(a - b) = a^2 - ba + ba - b^2 = a^2 - b^2$. Thus,

- $a^2 - b^2 = (a + b)(a - b)$

The difference between two perfect squares can always be factorised in this way. It can be useful to look for the difference between two perfect squares in an expression since, when factorised as above, the expression may simplify.

Example

Simplify $\dfrac{a^2 - b^2}{a + b}$

$$a^2 - b^2 = (a + b)(a - b)$$

So, $$\frac{a^2 - b^2}{a + b} = \frac{\cancel{(a + b)}(a - b)}{\cancel{a + b}}$$

$$\underline{= a - b}$$

Notice that we treat the quantity (a + b) as if it were a single term when we cancel it - we could not cancel just the a or just the b.

Check - Let a = 3 and b = 2:

- original expression: $(a^2 - b^2)/(a + b) = (3^2 - 2^2)/(3 + 2) = (9 - 4)/5 = 5/5 = 1$
- final expression: $a - b = 3 - 2 = 1$

The answers agree, as they should.

Example

Simplify $\dfrac{y^2 - 9}{y - 3}$

Here we need to see that $9 = 3^2$, and so:

$$\frac{y^2 - 9}{y - 3} = \frac{y^2 - 3^2}{y - 3}$$

$$= \frac{(y + 3)(y - 3)}{y - 3}$$

$$\underline{= y + 3}$$

Factorising a quadratic expression

- A quadratic expression is one in which the highest power of a variable is 2

Consider: $(2x + 1)(x + 3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$

The result that we get, $2x^2 + 7x + 3$, is a quadratic expression.

To factorise a quadratic expression, we need to do the *opposite* of the above, i.e. we start with a quadratic like $2x^2 + 7x + 3$ and turn it into a product of its factors such as $(2x + 1)(x + 3)$. So, in general we want to:

- turn a quadratic into the product of two factors looking like: $(ax \pm b)(cx \pm d)$

The a, b, c and d are numbers, and $\pm$ means 'plus or minus'.

If we consider how the expression $2x^2 + 7x + 3$ came from $(2x + 1)(x + 3)$, this gives us an idea of how to reverse the process.

In the expression $2x^2 + 7x + 3$, the first term, $2x^2$, comes from the product of two factors and the last term, +3, comes from the product of another two factors. So we list the possible factors for each of them:

factors of $2x^2$	factors of 3
$2x$ x	$+3$ $+1$ $+1$ $+3$ -3 -1 -1 -3

Now we try each possibility, and see which one gives the correct *middle* term, +7x:

- $(2x + 3)(x + 1) = 2x^2 + 2x + 3x + 3 = 2x^2 + 5x + 3$
- $(2x + 1)(x + 3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$ ← **correct**
- $(2x - 3)(x - 1) = 2x^2 - 2x - 3x + 3 = 2x^2 - 5x + 3$
- $(2x - 1)(x - 3) = 2x^2 - 6x - x + 3 = 2x^2 - 7x + 3$

Hence, $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

All the possibilities have been included above, for completeness, but it is not necessary to continue beyond the correct answer.

Part 2: ALGEBRA - contents

Algebraic Expressions and Relationships (Chapters 7 to 9)

Chapter 8

- ALGEBRAIC FRACTIONS
- EQUATIONS
- SIMULTANEOUS EQUATIONS
- FORMULAE

ALGEBRAIC FRACTIONS - contents

These are fractions which contain algebraic quantities, rather than just numbers

Simplifying algebraic fractions sometimes means factorising the numerator and/or the denominator. The last two examples both required algebraic fractions to be simplified by factorising the numerator.

Multiplication and division

- To multiply two numerical fractions we can multiply the numerators together and the denominators together and then see if we can simplify the result

For example,

$$\frac{3}{4} * \frac{2}{5} = \frac{6}{20} = \frac{3}{10}$$

We may prefer, if possible, to do some simplifying before multiplying the numerators and denominators:

$$\frac{3}{{}_2\cancel{4}} * \frac{\cancel{2}^1}{5} = \frac{3}{2} * \frac{1}{5} = \frac{3}{10}$$

- When multiplying two algebraic fractions, the process is the same:

$$\frac{2x}{y} * \frac{3y}{4z} = \frac{6\,xy}{4yz} = \frac{3\,x}{2\,z}$$

- To divide by a numerical fraction, we can invert the fraction and then multiply by it

For example,

$$1/2 \div 3/4 = \frac{1}{2} * \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$$

- We can do the same with algebraic fractions, for example:

Simplify $\dfrac{2xy^3}{3by} \div \dfrac{4y^2}{b}$

$$\frac{2xy^3}{3by} \div \frac{4y^2}{b} = \frac{2xy^3}{3by} * \frac{b}{4y^2} = \frac{2bxy^3}{12by^3} = \frac{x}{6}$$

We could have done some partial cancellation before multiplying, but the final result would have been the same.

Addition and subtraction

To add or subtract two fractions we:

- find the Lowest Common Multiple (LCM) of the denominators (the bottom parts), then
- make all the denominators equal to the LCM , changing each numerator, as required, to keep each fraction the same, then
- add or subtract the numerators, then simplify if possible

The LCM of algebraic quantities can often be found quite easily 'by inspection', i.e. just by comparing them.

For example, what is the LCM of: $2x^2 y^2$, $3x^3 y z$ and $x y z^4$

To find the LCM of all the expressions we *multiply the LCM of the coefficients by the highest power of each letter* :

- the LCM of 2 and 3 is 6
- the highest power of x is x^3
- the highest power of y is y^2
- the highest power of z is z^4

So, the LCM of $2x^2 y^2$, $3x^3 y z$ and $x y z^4 = 6 x^3 y^2 z^4$

Example

Simplify: $\dfrac{3x}{y^2} + \dfrac{x^2}{2yz}$

By comparing the denominators we see that their LCM is $2y^2z$

We want $2y^2z$ to be the common denominator, i.e. the same for both fractions.

To change y^2 to $2y^2z$ we multiply by $2z$, and we must do the same to $3x$ on top, giving $6xz$
To change $2yz$ to $2y^2z$ we multiply by y, and we must do the same to x^2 on top, giving x^2y.

So, we get:

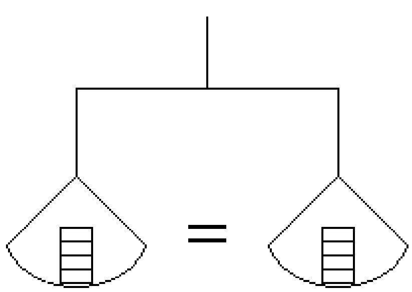
$$\frac{3x}{y^2} + \frac{x^2}{2yz} = \frac{6xz}{2y^2z} + \frac{x^2y}{2y^2z} = \frac{6xz + x^2y}{2y^2z} = \frac{x(6z + xy)}{2y^2z}$$

EQUATIONS - start of this chapter - contents

An equation states that two quantities, the left-hand side and the right-hand side, are equal.

- 3*4 + 1 = 13 is an arithmetic equation
- 2y = 4 is an algebraic equation
- 4y +7 = 3x -1 is an algebraic equation

An equation can be compared to a pair of balanced scales:



Suppose that the above scales are balanced with 4 grams on each side. The scales will remain balanced if we do the same to both sides. They remain balanced:

- if we add the same to each side - for example, add 3 grams to both
- if we subtract the same from both sides - for example remove 2 grams from both
- if we multiply both sides by the same - for example, treble both to make them 12 grams
- if we divide both sides by the same - for example, halve both to make them 2 grams

In algebraic equations we usually have one or more unknown quantities.

- *solving* an equation means finding values of the unknown quantities which make the equation true

Since both sides of an equation are equal, they will remain equal if we do the same to each side - just like the above scales. So, both sides of an equation will remain equal:

1. if we add the same to both sides
2. if we subtract the same from both sides
3. if we multiply both sides by the same
4. if we divide both sides by the same

We can use the above operations to solve many equations, i.e. to find the values of unknown quantities. This is illustrated below.

Linear equations

- In a linear equation, the highest power of a variable is 1

For example, $y = 2x + 1$ and $3x - 4 = 7x$ are linear equations. (Remember that x is the same as x^1)

Example - requiring addition to both sides

Solve $x - 3 = 4$

At the moment, the 3 is *subtracted* from x, so to remove it we *do the opposite* - we *add* 3 to both sides:

$$
\begin{aligned}
& x - 3 = 4 \\
\text{So,}\quad & x - 3 + 3 = 4 + 3 \\
& x = 4 + 3 \\
\text{So,}\quad & \underline{x = 7}
\end{aligned}
$$

Shortcut - you can go from the 1st to the 3rd line simply by taking the -3 across the equals sign and changing its sign to +3.

Check : When we have a solution, it can be substituted back into the original equation as a check:

- $x - 3 = 7 - 3 = 4$, which is correct

Example - requiring subtraction from both sides

Solve $x + 5 = 2$

At the moment, the 5 is *added* to x, so to remove it we do the opposite - we *subtract* 5 from both sides:

$$
\begin{aligned}
& x + 5 = 7 \\
\text{So,}\quad & x + 5 - 5 = 7 - 5 \\
& x = 7 - 5 \\
\text{So,}\quad & \underline{x = 2}
\end{aligned}
$$

Shortcut - you can go from the 1st to the 3rd line simply by taking the +5 across the equals sign and changing its sign to -5.

- *Check* : x + 5 = 2 + 5 = 7, which is correct

Example - requiring multiplication of both sides

Solve $\frac{x}{3} = 2$

At the moment, x is *divided* by 3, so to remove it we do the opposite - we *multiply* both sides by 3:

$$\frac{x}{3} = 2$$

So, $$3 * \frac{x}{3} = 2*3$$

$$x = 2*3$$

So, $$\underline{x = 6}$$

Shortcut - you can go from the 1st to the 3rd line simply by moving the 3 from the bottom on the left to the top on the right.

- Check: x/3 = 6/3 = 2, which is correct

Example - requiring division of both sides

Solve 4x = 8

At the moment, x is *multiplied* by 4, so to remove it we do the opposite - we *divide* both sides by 4:

$$4x = 8$$

So, $$\frac{4x}{4} = \frac{8}{4}$$

$$x = \frac{8}{4}$$

So, $$\underline{x = 2}$$

Shortcut - you can go from the 1st to the 3rd line simply by moving the 4 from the top on the left to the bottom on the right

- *Check* : 4x = 4*2 = 8, which is correct

Example

Solve 2x - 4 = 2

The following shows every step:

$$
\begin{array}{lll}
 & 2x - 4 = 2 & \\
\text{So,} & 2x - 4 + 4 = 2 + 4 & \text{(adding 4 to both sides)} \\
 & 2x = 2 + 4 & \\
\text{So,} & 2x = 6 & \\
\text{So,} & \frac{2x}{2} = \frac{6}{2} & \text{(dividing both sides by 2)} \\
 & x = \frac{6}{2} & \\
\text{So,} & \underline{x = 3} &
\end{array}
$$

We now repeat the process, but make use of a couple of the shortcuts mentions earlier - it makes the process simpler:

$$
\begin{array}{lll}
 & 2x - 4 = 2 & \\
 & 2x = 2 + 4 & \text{(taking the } -4 \text{ across and changing its sign)} \\
\text{So,} & 2x = 6 & \\
 & x = \frac{6}{2} & \text{(taking the 2 from the top on the left to the bottom on the right)} \\
\text{So,} & \underline{x = 3} &
\end{array}
$$

- *Check* : 2x - 4 = 2*3 - 4 = 6 - 4 = 2, which is correct

SIMULTANEOUS EQUATIONS - start of this chapter - contents

Consider:

$$3x + 2y = 7$$
$$x + y = 4$$

If we want values for the variables x *and* y which satisfy *both* equations, we call these *simultaneous solutions* and the equations are called *simultaneous equations* .

The solutions will look like x = something, and y = something. Hence, we need to get x by itself and then y by itself (or the other way round).

We can use a procedure called an *elimination method* :

$$3x + 2y = 7 \qquad (1)$$
$$x + y = 4 \qquad (2)$$

Suppose we want to eliminate y.
To do this we firstly make the term containing y the same in both equations.
We can do this my multiplying all of equation (2) by 2:

$$3x + 2y = 7 \qquad (3)$$
$$2x + 2y = 8 \qquad (4)$$

We now have 2y in both equations, so if we subtract the left side of (4) from the left of (3) the 2y will vanish - we also must subtract the right side of (4) from the right of (3), so:

$$3x + 2y - (2x + 2y) = 7 - 8$$
$$3x + 2y - 2x - 2y = -1$$
$$\underline{x = -1}$$

Now that we have x, we can substitute into one of the previous equations to find y:

$$x + y = 4$$
So, $-1 + y = 4$
So, $y = 4 + 1$
So, $\underline{y = 5}$

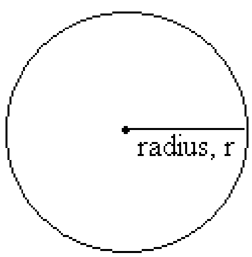
Check : The values obtained, x = -1 and y = 5, are supposed to satisfy *both* the original equations, so we substitute them into both to check that they do:

- (1) $3x + 2y = 3*(-1) + 2*5 = -3 + 10 = 7$ (correct)
- (2) $x + y = -1 + 5 = 5 - 1 = 4$ (correct)

FORMULAE (plural of 'formula') - start of this chapter - contents

- A formula is an equation which represents the relationship between particular quantities

For example, the area A of a circle of radius r, is given by:



$\boxed{\text{area of a circle, } A = \pi r^2}$

π (said as 'pie') $= 3.14159....$ (it may be on your calculator)

$= \frac{22}{7}$ (approximately)

Eg. If $r = 5\text{cm}$, then $A = \pi r^2 = \pi * 5^2 = 78.54 \text{ cm}^2$

Sometimes it is necessary to *transpose* a formula to makes a particular letter into the subject, i.e. to get the letter by itself on the left-hand side of the equation.

Example

Transpose the formula $A = \pi r^2$ to make r the subject.

Since we want r alone on the left, we could first simply reverse the equation:

$$\pi r^2 = A$$

Now divide both side by π:

$$\frac{\cancel{\pi} r^2}{\cancel{\pi}} = \frac{A}{\pi}$$

$$r^2 = \frac{A}{\pi}$$

Now take the square root of both sides:

$$\sqrt{r^2} = \sqrt{\frac{A}{\pi}}$$

$$\underline{r = \sqrt{\frac{A}{\pi}}}$$

Put $A = 78.54\text{cm}^2$, and show that $r = 5.0\text{cm}$ (as in the last calculation).

Part 2: ALGEBRA - contents

Algebraic Expressions and Relationships (Chapters 7 to 9)

Chapter 9

- QUADRATIC EQUATIONS
- FINDING A SOLUTION BY TRIAL AND IMPROVEMENT
- INEQUALITIES
- FLOW CHARTS

QUADRATIC EQUATIONS - contents

- A *polynomial equation* contains powers of a variable. For example:
  - $2x + 1 = 3$, $2x^2 + 7x = 4$, $2x^4 + 7x = 20$
- A *quadratic equation* is one in which the highest power of the variable is 2. For example:
  - $2x^2 + 7x + 3 = 0$

In general we can represent a quadratic equation by the *standard form* :

- $ax^2 + bx + c = 0$, where a, b and c are numbers (in the above example, a = 2, b = 7 and c = 3)

Other quadratic equations can be rearranged to look like this:

- Suppose that $2x^2 + 7x + 3 = 2$
- We could rearrange this as $2x^2 + 7x + 3 - 2 = 0$, or $2x^2 + 7x + 1 = 0$, so it now looks like the general equation above, with a zero on the right

We can always make the coefficient 'a', of x^2 , positive. For example:

- Suppose that $-3x^2 + x - 2 = 0$
- If we multiply both sides (*every* term) by -1, then we get: $+3x^2 - x + 2 = 0$. The right hand side stays zero, since anything multiplied by zero

equals zero

Solving a quadratic by factorising

Recall that to *solve* an equation means finding the value(s) of the unknown quantity which make the equation true.

If we had the quadratic equation: $x^2 = 4$, then:

- one solution is $x = 2$, since $2^2 = 4$
- but another solution is $x = -2$, since $(-2)^2 = 4$

It is generally the case that we expect two solutions to a quadratic equation.

From previous notes, we know that (some) quadratics can be expressed as a product of two brackets.

Suppose that $x^2 + x - 2 = 0$

We can factorise the expression on the left using the method already described to get:

- $x^2 + x - 2 = (x + 2)(x - 1)$

So, $(x + 2)(x - 1) = 0$.

Now, since *anything* multiplied by zero equals zero, then either $(x + 2) = 0$ or $(x - 1) = 0$.

- If $x + 2 = 0$, then $x = - 2$
- If $x - 1 = 0$, then $x = +1$

So, the solutions of $x^2 + x - 2 = 0$, are $x = -2$ and $x = 1$.

Check : Remember that the solutions of $x^2 + x - 2 = 0$ are those values of x that make the equation true. So we put each value into the equation as a check:

- Using $x = -2$: $x^2 + x - 2 = (-2)^2 - 2 - 2 = 4 - 4 = 0$
- Using $x = 1$: $x^2 + x - 2 = 1^2 + 1 - 2 = 2 - 2 = 0$

So, a *procedure for solving quadratic equations* is to:

1. factorise the equations into the product of two brackets
2. set each bracket in turn equal to zero, and rearrange to get x for each - these are the solutions
3. put the x values back into the original quadratic expression, to check that it equals zero

Solving a quadratic by formula

The standard form of writing a quadratic equation is:

- $ax^2 + bx + c = 0$, where a, b and c are numbers

We can show that the solution of this type of equation can be found using the formula:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Let' s apply this to the equation already solved: $x^2 + x - 2 = 0$

Here $a = 1$, $b = 1$, $c = -2$:

So,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4*1*(-2)}}{2*1}$$

$$= \frac{-1 \pm \sqrt{1 + 8}}{2}$$

$$= \frac{-1 \pm \sqrt{9}}{2}$$

$$= \frac{-1 \pm 3}{2}$$

So, $x = \frac{-1 + 3}{2} = \frac{2}{2} = 1$

Or, $x = \frac{-1 - 3}{2} = \frac{-4}{2} = -2$

So, $\underline{x = 1 \text{ or } x = -2}$

FINDING A SOLUTION BY TRIAL AND IMPROVEMENT - start of this chapter - contents

If $x^3 = 20$, what is the value of x, to 2 decimal places?

The idea of the *trial and improvement method* is to gradually get nearer and nearer to the correct solution. This is illustrated below for the above problem. We start with a low value of x and gradually increase it:

If x = 1, then $x^3 = 1$
If x = 2, then $x^3 = 8$
If x = 3, then $x^3 = 27$

We want to find the x value for $x^3 = 20$. Now, 20 is between 8 and 27.
So, the value of x for which $x^3 = 20$ must be between x = 2 and x = 3.

Let's try x = 2.5, which is between 2 and 3: If x = 2.5, then x^3 = 15.625
So, the value of x for which x^3 = 20 must be between x = 2.5 and x = 3

Let's try x = 2.7: If x = 2.7, then x^3 = 19.683
So, the value of x for which x^3 = 20 must be between x = 2.7 and x = 3

Let's try x = 2.8: If x = 2.8, then x^3 = 21.952
So, the value of x for which x^3 = 20 must be between x = 2.7 and x = 2.8

Let's try x = 2.71: If x = 2.71, then x^3 = 19.903
So, the value of x for which x^3 = 20 must be between x = 2.71 and x = 2.8

Let's try x = 2.72: If x = 2.72, then x^3 = 20.124
So, the value of x for which x^3 = 20 must be between x = 2.71 and x = 2.72

We see that the value of x^3 is nearer to 20 using x = 2.71 than when using x = 2.72, so the solution of x^3 = 20 to 2 decimal places is x = 2.71.

The method looks a lot simpler if we put the values in a table:

trial x	x^3	the x satisfying $x^3 = 20$ must be
1	1	greater than 1
2	8	greater than 2
3	27	between 2 and 3
2.5	15.625	between 2.5 and 3
2.7	19.683	between 2.7 and 3
2.8	21.952	between 2.7 and 2.8
2.71	19.903	between 2.71and 2.8
2.72	20.124	between 2.71 and 2.72

Note - the above type of procedure is also called an *iterative method* of finding a solution.

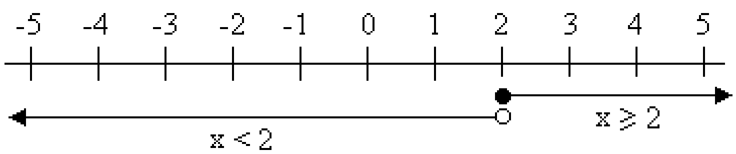
The answer in the above case could be found directly by using your calculator:

- If $x^3 = 20$, then $x = 20^{1/3}$ (= 2.714)

However, the method is valuable because it can be applied to problems which cannot be solved so directly.

INEQUALITIES - start of this chapter - contents

The following represents the *number line* - it extends forever in both directions. Any number lies somewhere on this line. If we pick one particular number, such as 2, then any number is either equal to 2 or is less than 2 or is greater than 2.



The empty circle ○ indicates that $x = 2$ is not included in the possible values of x

The solid circle ● indicates that $x = 2$ is included in the possible values of x

This introduces the idea of *inequalities* , which requires some new symbols:

$x < 2$ stands for 'x is less than 2'

$x > 2$ stands for 'x is greater than 2

$x \leq 2$ stands for 'x is less than or equal to 2'

$x \geq 2$ stands for 'x is greater than or equal to 2'

We can sometimes combine two inequalities into a single expression. For example:

$1 < x \leq 4$ means that 1 is less than x and x is less than or equal to 4
$-3 \leq x < 0$ means that -3 is less than or equal to x and x is less than zero

An inequality that contains an unknown variable, can be solved using the same rules for addition, subtraction, multiplication and division as already used with equations, but with one extra rule:

- if both sides of an inequality are multiplied or divided by a *negative number* then the inequality sign must be *reversed*

Example

Solve the inequality $4x - 1 \geqslant 2x + 11$

As with an equation, we want x alone on the left-hand side.
We add 1 to both sides, or equivalently, take the -1 across and change it to $+1$:

$$4x \geqslant 2x + 11 + 1$$

$$4x \geqslant 2x + 12$$

Now subtract 2x from both sides, or equivalently, take the $+2x$ across and change it to $-2x$:

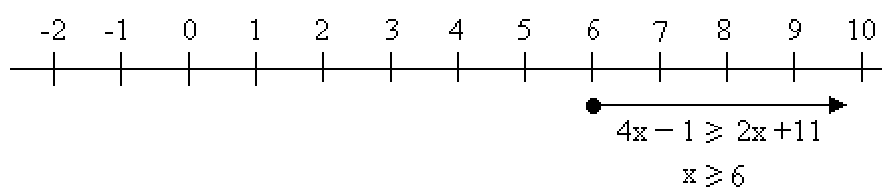
$$4x - 2x \geqslant 12$$

$$2x \geqslant 12$$

Now divide both sides by 2, or equivalently, take the 2 from the top on the left to the bottom on the right:

$$x \geqslant \frac{12}{2}$$

$$\underline{x \geqslant 6}$$



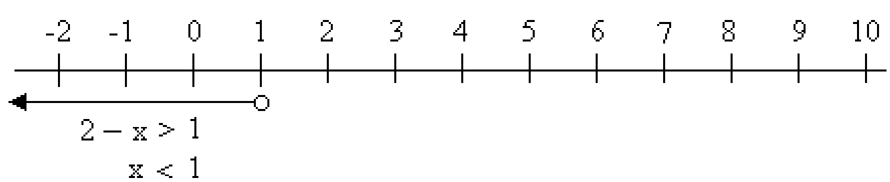
Example

Solve the inequality $2 - x > 1$

$$-x > 1 - 2$$

$$-x > -1 \qquad \text{(multiply by } -1\text{, and reverse the inequality)}$$

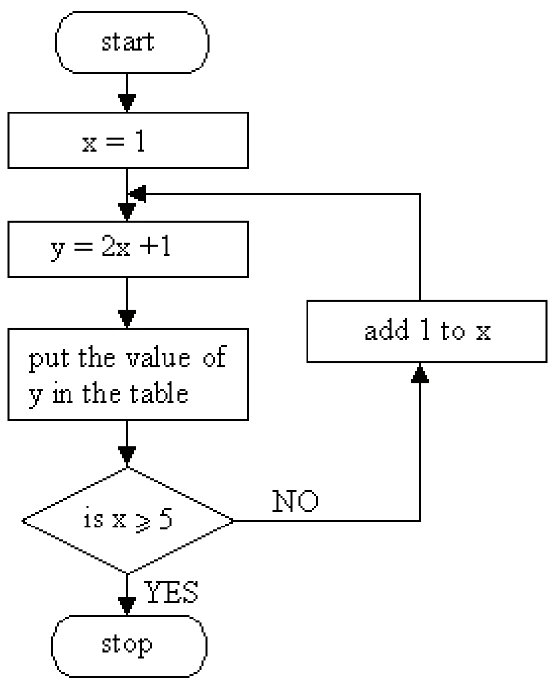
$$\underline{x < 1}$$



FLOW CHARTS (OR FLOW DIAGRAMS) - start of this chapter - contents

A flow chart is a way of representing a process as a sequence of actions.

The following flow chart describes the process of completing the table on the right:



x	1	2	3	4	5
$y = 2x + 1$					

The diamond shaped box is a *decision box* . The box asks a question and a 'YES' or 'NO' arrow is then followed, depending on the answer. In the above case, if x is greater than or equal to 5, then the YES is followed which leads to the process being stopped.

Part 2: ALGEBRA - contents

Graphs (Chapters 10 to 12)

Chapter 10

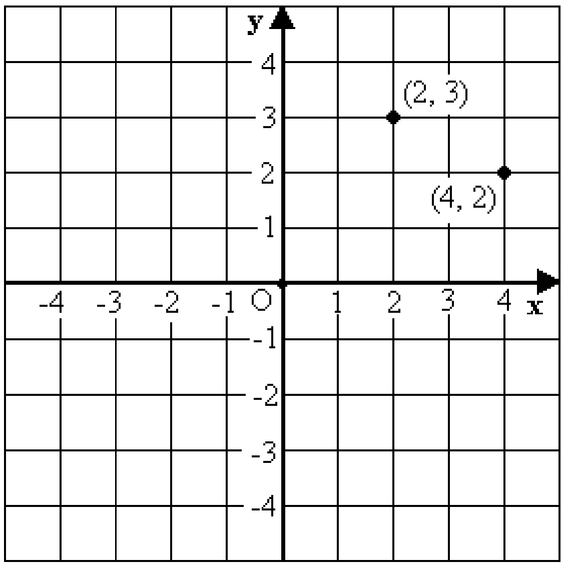
- CARTESIAN COORDINATES
- STRAIGHT LINE GRAPHS
- PARABOLAS
- CUBIC FUNCTIONS
- RECTANGULAR HYPERBOLA
- EXPONENTIAL FUNCTIONS

CARTESIAN COORDINATES - contents

- A *cartesian graph* consists of a grid on which we draw two straight lines at right angles
- These lines are called *rectangular axes* , one is called the x-axis and the other is the y-axis
- The positive x-axis is to the right and the positive y-axis is upwards
- The lines usually meet at the *origin* , where x = 0 and y = 0
- By numbering the axes, such as indicated below, the location of any point can be specified by its x and y *coordinates*

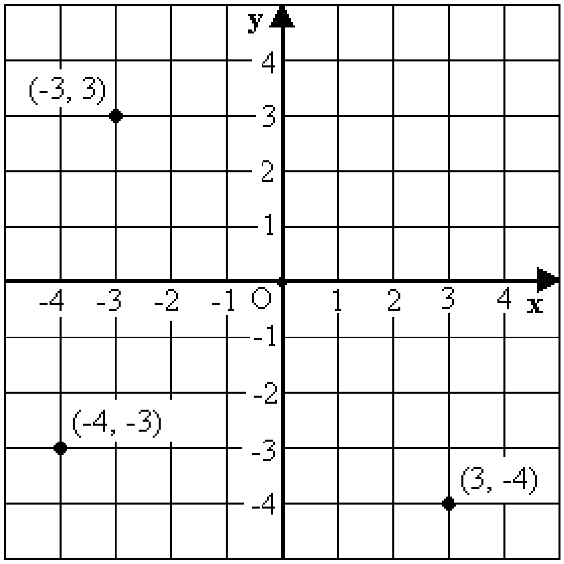
A pair of coordinates is represented by, for example, (4,2). The first number is an x value, and the second is a y value:

- the point (4, 2) is found by starting at the origin and moving 4 in the positive x direction (right), then 2 in the positive y direction (up)
- the point (2, 3) is found by starting at the origin and moving 2 in the positive x direction (right), then 3 in the positive y direction (up)



Coordinates can also be negative, for example:

- the point (3, -4) is found by starting at the origin and moving 3 in the positive x direction (right), then 4 in the negative y direction (down)
- the point (-4, -3) is found by starting at the origin and moving 4 in the negative x direction (left), then 3 in the negative y direction (down)
- the point (-3, 3) is found by starting at the origin and moving 3 in the negative x direction (left), then 3 in the positive y direction (up)



STRAIGHT LINE GRAPHS - start of this chapter - contents

Plotting a graph

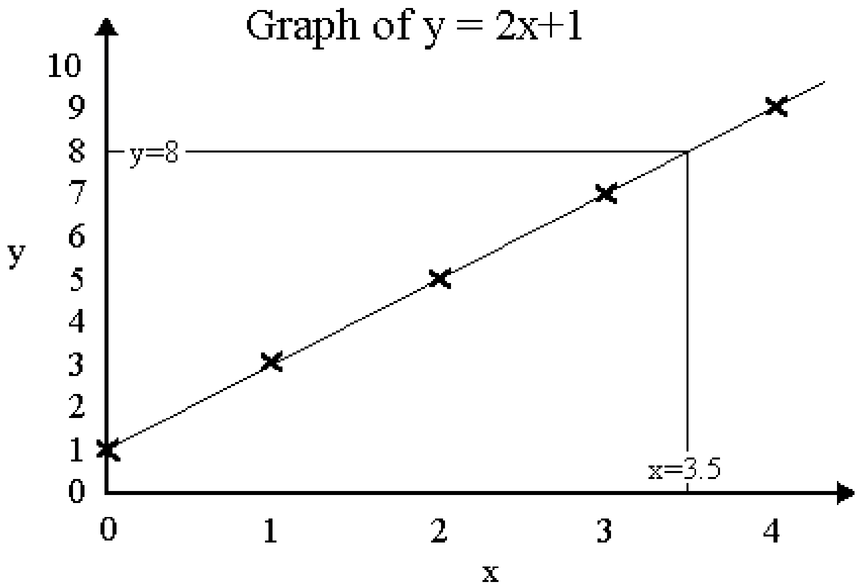
If you are given an equation such as **y - 2x -1 = 0** , and asked to plot (or draw) a graph of y against x, the first thing to do is to get y by itself on the left of the equal sign.

In the above case, we take the two terms, -2x and -1, to the right of the equal sign, changing the signs of both of them, and so we get the equation **y = 2x + 1** . In this form we say that y is expressed as *a function of x* . The value we get for y depends on the value we put in the equation for x.

We now choose a range of values for x and work out the corresponding values for y:

x	0	1	2	3	4
2x	0	2	4	6	8
y=2x+1	1	3	5	7	9

Now let's plot a graph of y (vertical) against x (horizontal):



When you plot a graph:

- chose the scales so that the graph is quite large - both axes should occupy at least half of the side of the page (if they do not, then you could have doubled the scale)
- label both axes, including units, if there are any
- make the individual points stand out, for example, by using crosses (x) - if you use small dots, they may disappear when you draw a line through them, which could cost marks, since an examiner will be looking to see if individual points have been plotted correctly
- give the graph a title

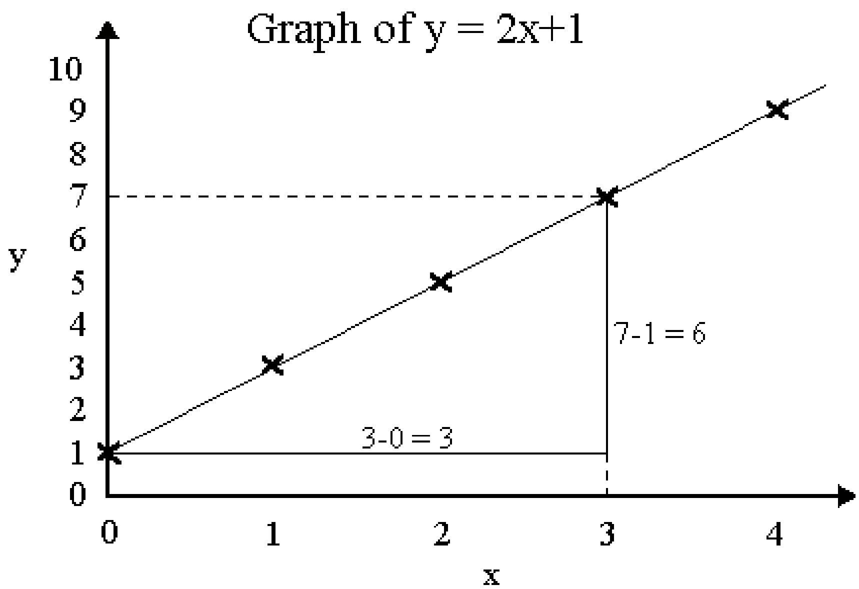
Once a graph is drawn, it can be used to find y for any chosen value of x:

- at the chosen x value, draw a vertical line, so that it meets the graph - this has been done at x = 3.5 on the above graph
- draw a horizontal line until it meets the y-axis - in the above case it meets at y = 8

So, when x = 3.5, y = 8. In this case, you could have got the answer directly from the equation y = 2x +1, but you may be given a graph without knowing its equation, so you need to know how to read values from a graph.

Gradient

The above graph is a straight line. The incline of a graph is called its *slope* or *gradient* . We can work out the gradient from the graph by drawing a triangle on it as below:



The triangle can be drawn anywhere but, for accuracy, the bigger it is the better. We calculate the gradient as:

$$\text{gradient} = \frac{\text{vertical side}}{\text{horizontal side}} = \frac{6}{3} = 2$$

In the graph, the y axis is drawn through the point x = 0, and we see that he graph cuts (or intercepts) the y axis at y = 1.

The equation for the graph is y = 2x + 1, and notice that:

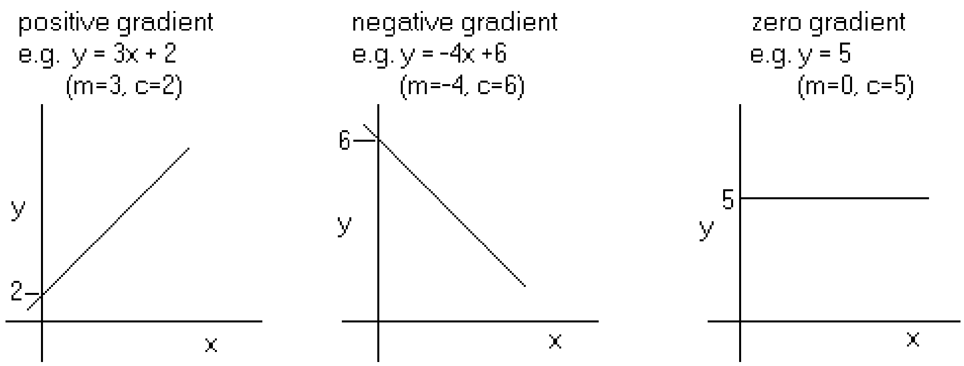
- the gradient equals the number in front of the x, and
- the intercept on the y axis is the number following the 2x

This pattern is *always* followed for straight line graphs. This leads to *the general equation for a straight line* :

$$\boxed{y = mx + c}$$

Where m = gradient of the graph, and c = intercept on the y axis.

The value of m, the gradient, can be positive, negative or zero:



Notice that to get the value of c directly from the equation y = mx + c, we put x = 0, then we have y = c.

Note: In the straight line equation, y = mx + c, If the value of c is changed, this changes the point at which the graph cuts the y axis, but it does not change its gradient (m). So:

- straight line graphs with the same m values, but different c values, are parallel

For example, the following produce parallel lines when plotted:

- y = 2x +1
- y = 2x + 4
- y = 2x + 10 etc.

Straight line graphs at right angles

Consider the functions y = 2x +1 and y = -0.5x +6. These both have the form y = mx +c, so will both produce straight line graphs. The first has a positive slope (+2), and the second has a negative slope (-0.5).

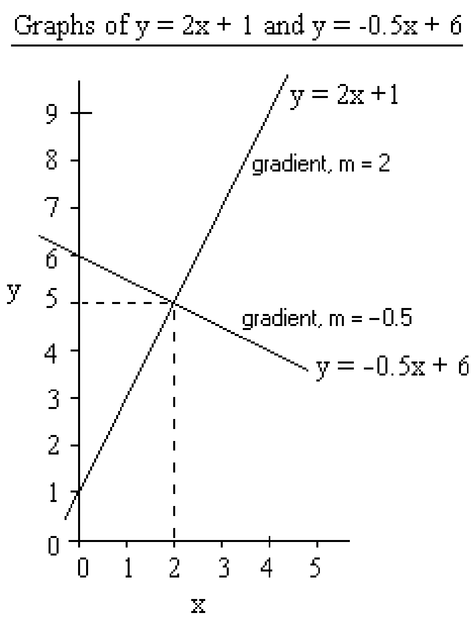
For y = 2x +1:

x	0	1	2	3	4
2x	0	2	4	6	8
y=2x+1	1	3	5	7	9

For y = -0.5x +6:

x	0	1	2	3	4
-0.5x	0	-0.5	-1	-1.5	-2
y = -0.5x + 6	6	5.5	5	4.5	4

The two sets of points are now plotted on the same axes:



We can see by eye that the lines appear to be at right angles (90 °) to each other.

Notice that if we multiply their gradients, we get 2*(-0.5) = -1. This is a general rule:

- If the product of the gradients of two straight line graphs equals -1, then the lines will be at right angles to each other when plotted on the same axes

Notice also, that plotting graphs for different functions of x on the same axes lets us find out where the graphs cross.

The above graphs cross at x = 2 and y = 5. Putting x = 2 into the two functions of x, we have:

- y = 2x +1 = 2*2 + 1 = 4 + 1 = 5
- y = -0.5x + 6 = -0.5*2 + 6 = -1 + 6 = 5

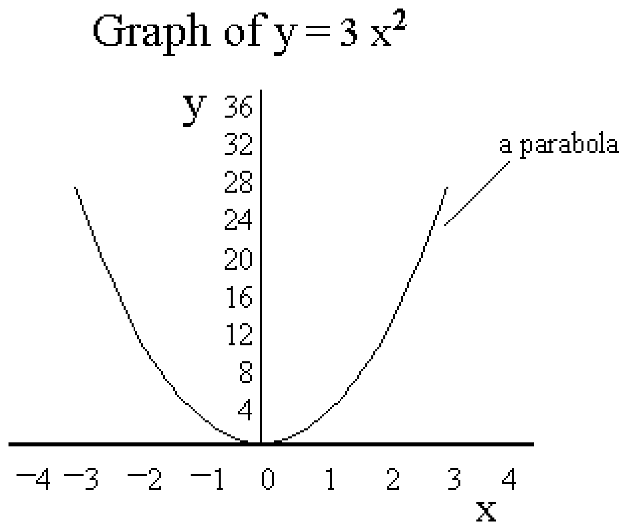
PARABOLAS - start of this chapter - contents

Lots of functions of x produce curved graphs, and a few are given special names.

Consider the quadratic equation $y = 3x^2$. To plot the graph, we work out y for a few selected values of x:

x	-3	-2	-1	0	1	2	3
$y = 3x^2$	27	12	3	0	3	12	27

From this we can plot the graph:

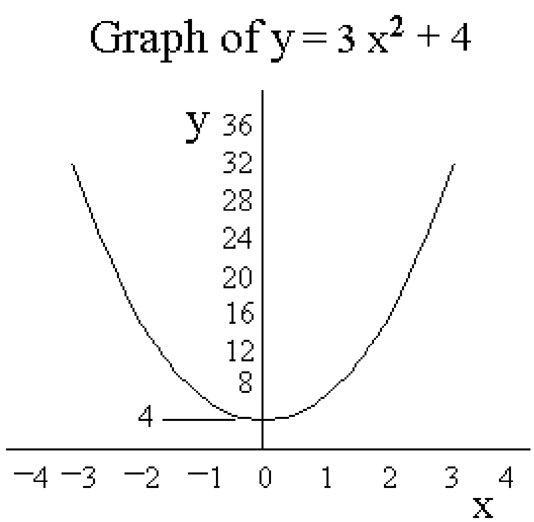


Note that if $y = -3x^2$, then all the y values in the above table would be negative, so the above graph would be the opposite way up.

Suppose that we change the original equation to $y = 3x^2 + 4$:

x	-3	-2	-1	0	1	2	3
$y = 3x^2 + 4$	31	16	7	4	7	16	31

The effect of the +4 is to move the previous graph by 4 in the +y direction



To be general:

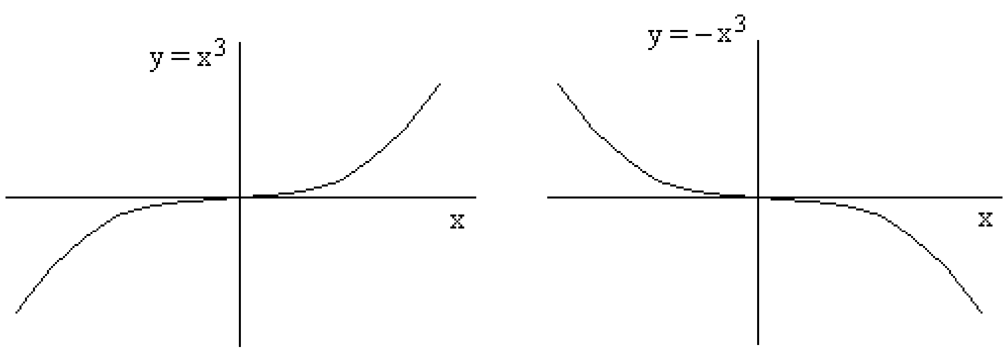
- A quadratic equation is one which can be written like $y = ax^2 + bx + c$, where a, b and c are numbers, and *a is not zero* , and these produce graphs called parabolas (we specify that a is not zero because, if it were zero, then ax^2 would vanish and we would have a straight line equation $y = bx + c$)

Note: Given an equation, it often helps to rearrange it into the form y = ... to help decide what shape graph it would produce. For example, if you are asked to plot or describe the shape of the graph for $3x^2 - y + 4 = 0$, it may help to firstly rearrange it as $y = 3x^2 + 4$, then it can be seen to be like $y = ax^2 + c$, and so it will be a parabola.

CUBIC FUNCTIONS - start of this chapter - contents

A cubic function of x is one in which the highest power of x is 3. The following are all cubic equations:

- $y = 4x^3 + x^2 - 2x$
- $y = 3x^3 + 7x + 5$
- $y = x^3$



To be more general:

- A cubic equation is one which can be written as $y = ax^3 + bx^2 + cx + d$, where a, b c and d are numbers and a is not zero, and these produce graphs similar to the above (we specify that a is not zero because, if it were zero, then ax^3 would vanish and we would have a quadratic equation $y = bx^2 + cx + d$)

RECTANGULAR HYPERBOLA - start of this chapter - contents

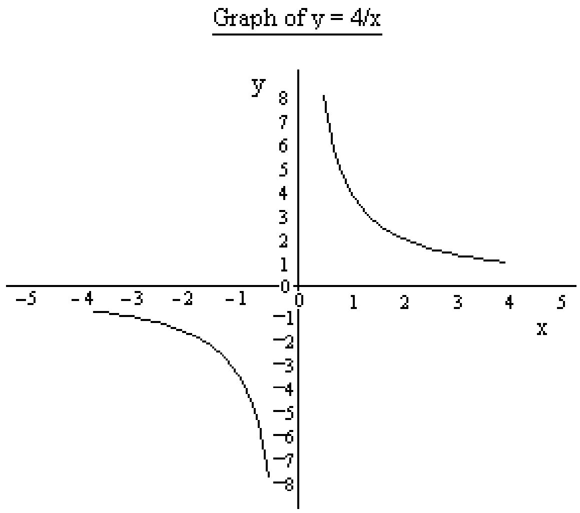
- An equation which can be arranged to look like y = a/x, where a is a number, but not zero, produces a rectangular hyperbola

Again, you may need to rearrange an equation to get y alone to see if it looks like y = a/x.

For example, suppose that xy = 4. This rearranges to y = 4/x. We can choose a set of x values, work out the corresponding y values, and then plot y against x to obtain the typical shape of a rectangular hyperbola.

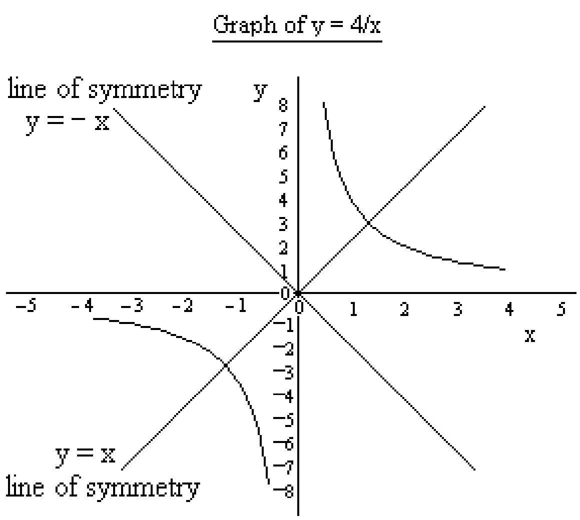
Note that when working out values for the table, we cannot include x = 0. Try 4/0 on a calculator. You get an error message since anything divided by zero equals infinity.

x	-4	-3	-2	-1	1	2	3	4
y = 4/x	-1	-1.33	-2	-4	4	2	1.33	1



Notice that the curves never meet or cut the axes. As x gets close to zero, the curves head towards infinity.

Also, notice that the curves have two lines of symmetry (we return to the topic of symmetry latter):

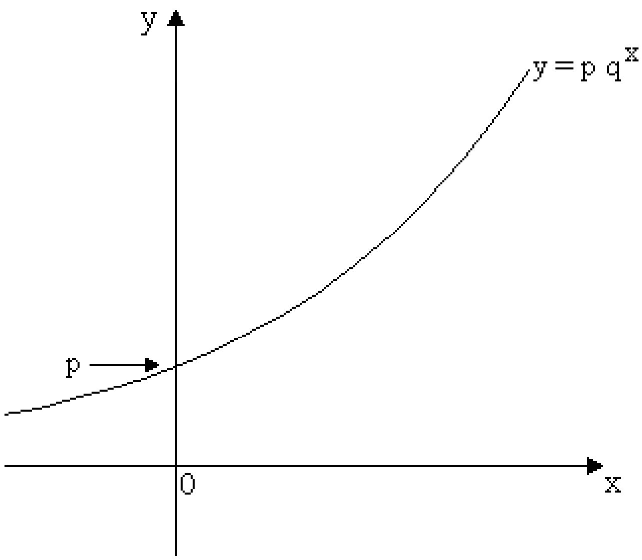


EXPONENTIAL FUNCTIONS - start of this chapter - contents

An exponential function of x has the form **$y = p\, q^x$** (only q is raised to the power x, *not* p). In this equation:

- p is a constant (i.e. its value does not change), and
- q is a positive constant

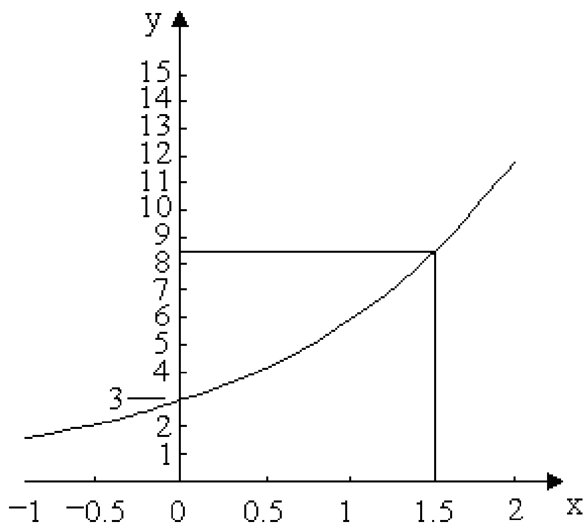
Notice that if we put x = 0, then y = p, since $q^0 = 1$ (since any number to the power zero equals 1)
So in a graph of y against x, when x = 0, the graph cuts the y axis at y = p:



Given a graph such as the above, we can find both p and q, as in the next example:

Example

It is believed that the relationship between y and x is of the form: $y = p\,q^x$, where p and q are constants.
Use the graph below to determine values for p and q.



Firstly, when x = 0, y = p = intercept on y axis = 3.0

Now that we have p, we can choose any pair of values on the graph to determine q:

For x = 1.5, we see that y = 8.5, so:

$$8.5 = 3.0 * q^{1.5}$$

So, $q^{1.5} = 8.5/3.0 = 2.83$

So, $q = (2.83)^{1/1.5} = 2.0$

On the calculator, you would enter:

- 2.83
- press y^x (or equivalent)
- press (
- enter 1/1.5
- press)
- press =

You should get, q = 2.0

So, the original equation can be expressed as: $y = 3.0 * 2.0^x$

As a check:

- when x = 0, $y = 3 * 2.0^0 = 3 * 1 = 3$
- when x = 1.5, $y = 3 * 2.0^{1.5} = 3 * 2.83 = 8.5$

Part 2: ALGEBRA - contents

Graphs (Chapters 10 to 12)

Chapter 11

- SOLVING EQUATIONS USING GRAPHS
- EQUATIONS CHANGED TO LINEAR GRAPHICAL FORM
- TRANSFORMATION OF FUNCTIONS/GRAPHS
- GRAPHS AND INEQUALITIES

SOLVING EQUATIONS USING GRAPHS - contents

A quadratic equation

When we 'solve' and equation such as $x^2 - 4 = 0$, this means that we find the value(s) of x which make the equation true.

We can solve the above equation by isolating the x by itself on the left of the equal sign. Thus,

- $x^2 - 4 = 0$, so
- $x^2 = 4$ (taking the -4 across and changing its sign), so
- $x = +2$ or -2

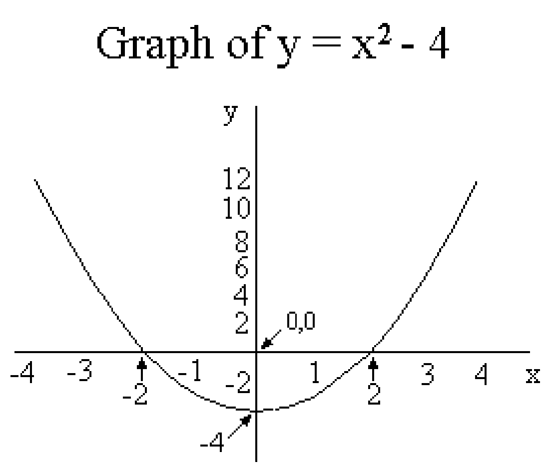
We can also solve this (and more complex equations) by drawing a graph.

- We let $y = x^2 - 4$. When $y = 0$, we get back the original equation, $0 = x^2 - 4$, so
- We plot a graph of y against x for a range of x values, and
- The x values on the graph at which $y = 0$ are the solutions of the equation $0 = x^2 - 4$

So, we first produce a table of x and y values:

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2 - 4$	12	5	0	-3	-4	-3	0	5	12

Then we plot the graph, drawing a smooth curve through the points:



The points where the graph cuts the x axis are the x values at which y = 0. And since $y = x^2 - 4$, these values of x are the solution of the equation $0 = x^2 - 4$. We see that these points are at x = 2 and x = -2.

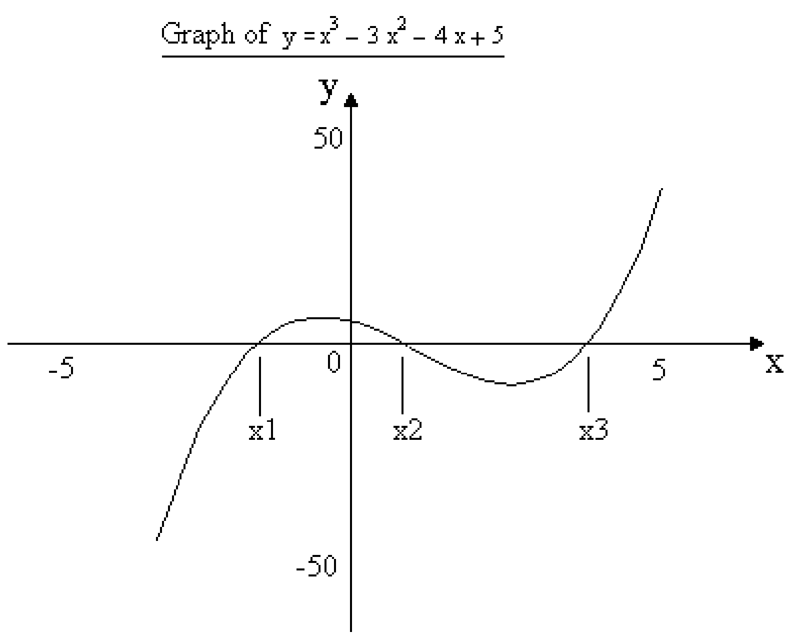
A cubic equation

Suppose that we want the solution of: $x^3 - 3x^2 - 4x + 5 = 0$.

We can do this the same way:

- We let $y = x^3 - 3x^2 - 4x + 5$. When y = 0, we get the original equation, so
- We plot a graph of y against x for a range of x values, and
- The x values on the graph at which y = 0 are the solutions of the equation $0 = x^3 - 3x^2 - 4x + 5$

The graph is shown below, and we see that there are 3 solutions (labelled x1, x2 and x3):

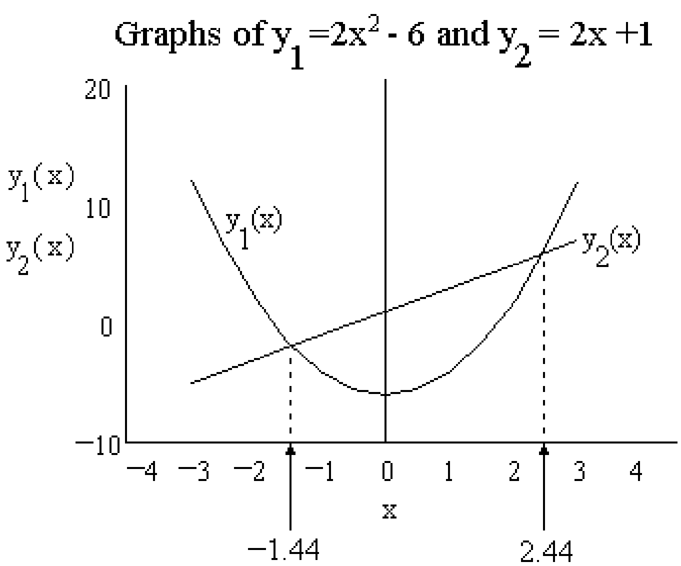


Simultaneous equations

Suppose that we have two functions of x, $y_1 = 2x^2 - 6$ and $y_2 = 2x + 1$ and we want to know which values of x make y_1 equal to y_2 , i.e. we want the *simultaneous solutions* of both equations.

Procedure:

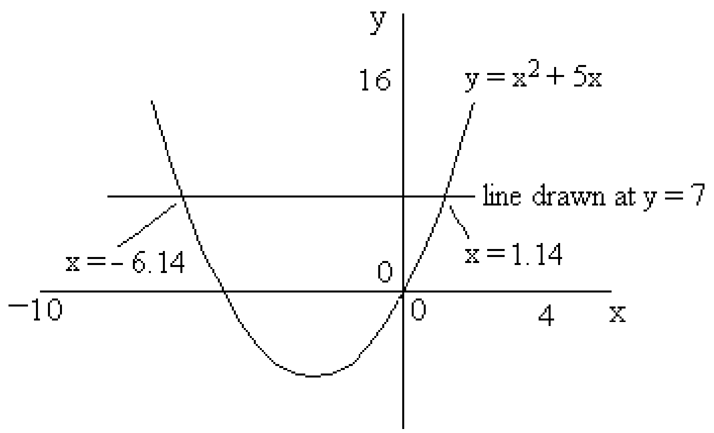
- Work out y_1 and y_2 for the same range of x values
- Plot both graphs on the same axes
- The points at which the graphs *cross* give the values of x which make y_1 and y_2 the same



Example

Use the graph of $y = x^2 + 5x$ to solve $x^2 + 5x = 7$.

Firstly we plot a graph of $y = x^2 + 5x$. Then we draw a horizontal line at y = 7.



The solutions are where the graph and the horizontal line cross: at x = -6.14 and x =1.14

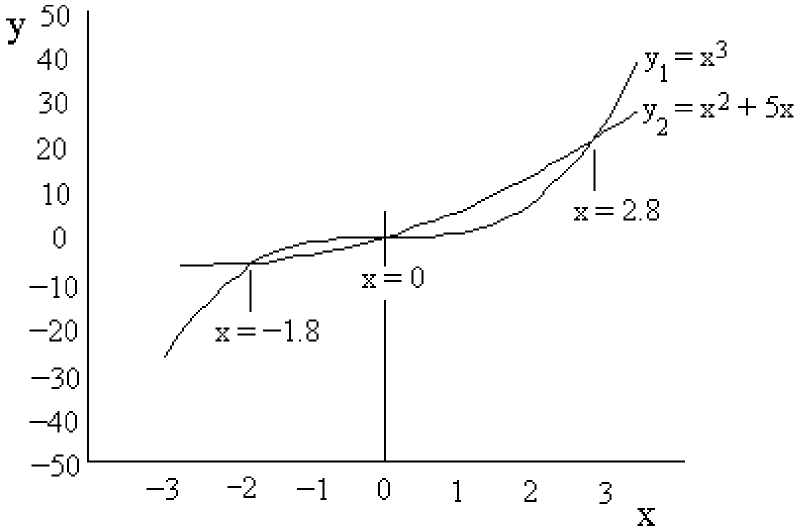
Example

Solve graphically the equation $x^3 = x^2 + 5x$.

Here we can plot graphs for:

- $y_1 = x^3$,and
- $y_2 = x^2 + 5x$

The x values at which the graphs cross are the solutions of $x^3 = x^2 + 5x$, since when they cross, $y_1 = y_2$.



The graphs cross three times, giving the solutions x = -1.8, x = 0, and x = 2.8

EQUATIONS CHANGED TO LINEAR GRAPHICAL FORM - start of this chapter - contents

We've seen that for an equation of the form y = m x + c, that:

- a graph of y against x is a straight line
- m = the slope (gradient) of the graph
- c = intercept on y axis (where x = 0)

Now consider, for example, $y = 3x^2 + 10$. If we plot y against x, we will get a curve (a parabola).
However, compare this equation with y = mx + c.

$$y = 3 x^2 + 10$$
$$y = m x + c$$

These equation have the same form: **a variable = a constant * a variable + a constant** . Further:

- if we plot y against $\mathbf{x^2}$ (instead of x) we get a straight line of gradient 3 and intercept 10 on the y axis

This is the case for any equation that can be arranged to 'look' like y = m x + c. We can plot y against whatever corresponds to x to get a straight line graph.

Example

It is believed that y and x are related by an equation of the form: $y = a x^2 + b$. Use the following data to determine values for a and b:

x	1	2	3	4	5
y	6.9	12.9	23.1	37.1	55.0

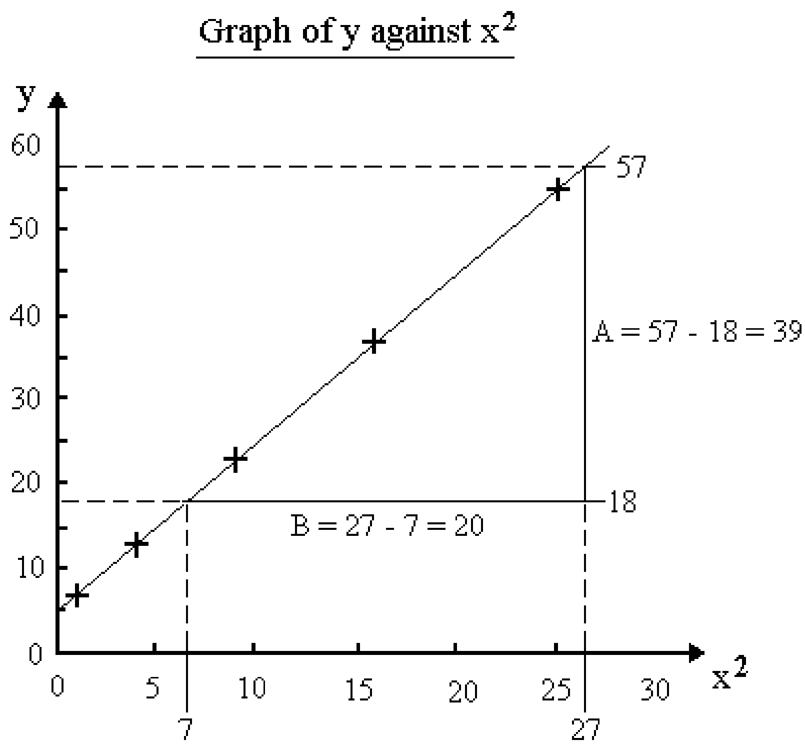
If $y = a x^2 + b$ then a graph of y against x will be a curve.

However, if we compare $y = a x^2 + b$ with $y = m x + c$, then plotting y against x^2 should produce a straight line graph of *slope a* , and *intercept b* on the y axis.

We need to add values of x^2 to the above data:

x	1	2	3	4	5
y	6.9	12.9	23.1	37.1	55.0
x^2	1	4	9	16	25

We plot the graph, and draw a best fit straight line through the points:



In working out the slope, we can draw the triangle anywhere. However, it can help to choose it so that B is an easy number to divide by. In the above case B = 20, so we can calculate the slope mentally (even if you use a calculator to find the slope, it is still useful to be able to do a quick mental check on the answer).

- slope, a $= \frac{A}{B} = \frac{39}{20} = 1.95 = 2.0$ (to 1 d.p.)
- intercept on y axis, b = 5.0

Now we can write the original equation as:

- $y = 2.0\ x^2 + 5.0$

As a check, choose a couple of x values from the data, and see if this equation gives good values for y:

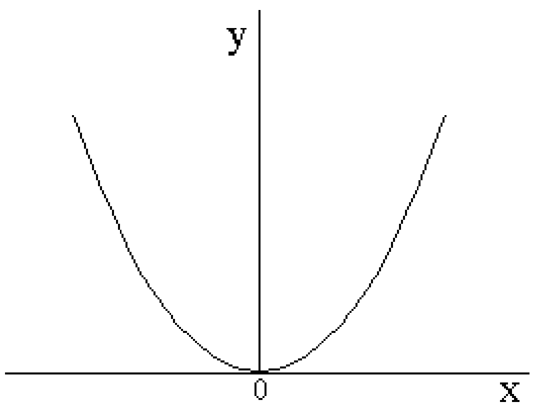
- $x = 1$, $y = 2.0*1^2 + 5.0 = 2 + 5 = 7$ (value in the table is 6.9)
- $x = 5$, $y = 2.0*5^2 + 5.0 = 5 + 5 = 55$ (value in the table is the same)

TRANSFORMATION OF FUNCTIONS/GRAPHS - start of this chapter - contents

An expression such as $2x^2 + 1$ is called a function of x.

We may write these as $f(x) = 2x^2 + 1$ or often as $y = 2x^2 + 1$.

Consider the function $y = f(x) = x^2$. This is a quadratic equation, and we've see previously that it produces a U-shaped graph, called a parabola:



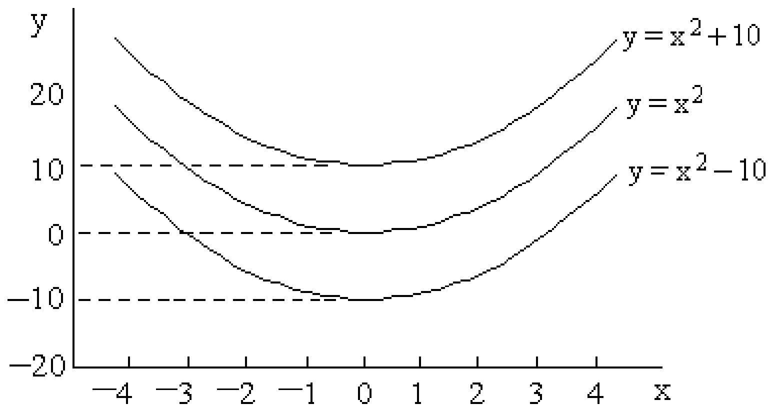
Now, if $y = f(x) = x^2$ has the above shape, what about:

1. $y = f(x) + a = x^2 + a$, or
2. $y = f(x + a) = (x + a)^2$, or
3. $y = f(kx) = (kx)^2$, where a and k are constants

1. Consider:

- $y = f(x) = x^2$
- $y = f(x) + 10 = x^2 + 10$, and
- $y = f(x) - 10 = x^2 - 10$

Plotting all three on one pair of axes we can compare them:

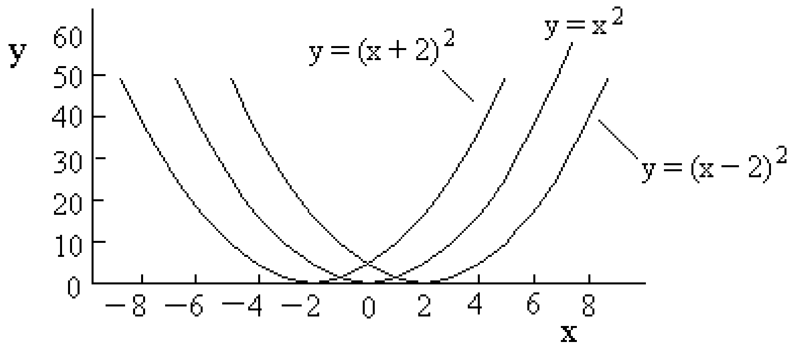


We see that:

- $f(x) + 10$ is 10 above $f(x)$, and
- $f(x) - 10$ is 10 below $f(x)$

2. Consider:

- $y = f(x) = x^2$
- $y = f(x + 2) = (x + 2)^2$, and
- $y = f(x - 2) = (x - 2)^2$

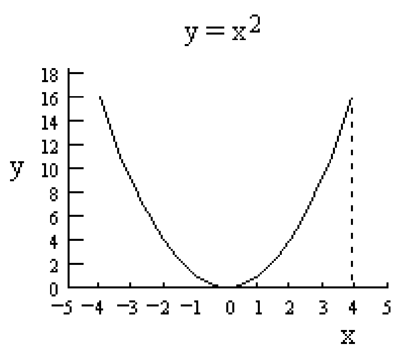


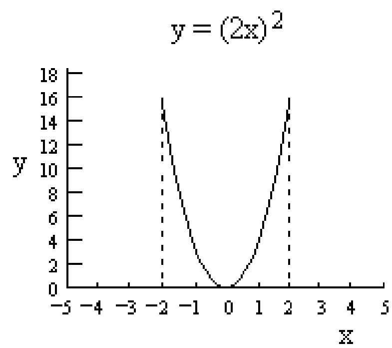
We see that:

- f(x + 2) is 2 to the left of f(x)
- f(x - 2) is 2 to the right of f(x)

3. Consider:

- y = f(x) = x^2
- y = f(2x) = $(2x)^2$





Note that:

- when x = 4, f(x) = 4^2 = 16
- when x = 2, f(2x) = $(2x)^2$ = $(2*2)^2$ = 4^2 = 16

So, $f(2x) = (2x)^2$ is half the width of $f(x) = x^2$.

So, transforming f(x) in one of the above ways:

- moves the graph of f(x) up or down, or
- moves the graph of f(x) left or right, or
- changes the width of the graph of f(x)

GRAPHS AND INEQUALITIES - start of this chapter - contents

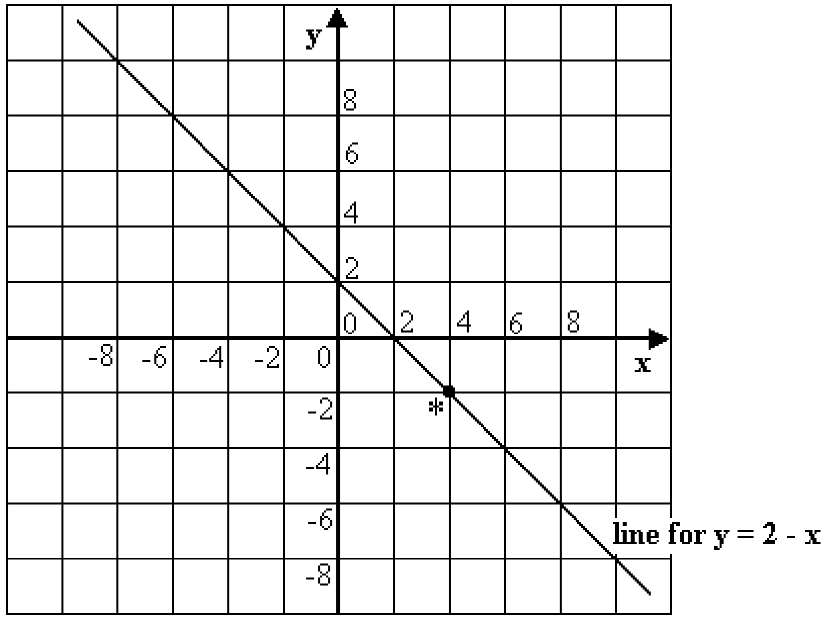
We consider the inequality: $y \leqslant 2 - x$

The solution to this is all those values of x and y which make the inequality true.

The solution can be represented graphically. Firstly we draw the graph of the *equation* y = 2 - x, for a range of x values:

x	-6	-4	-2	0	2	4	6
y = 2 - x	8	6	4	2	0	-2	-4

We now draw the line for y = 2 - x.

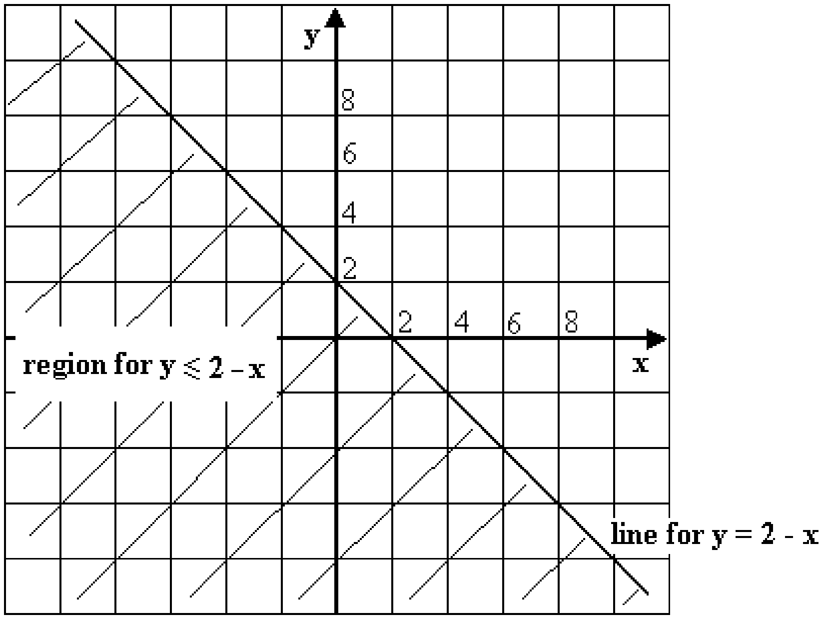


If we look at a point on the line, such as the * then:

- at that point y = 2 - x = 2 - 4 = -2
- every y value directly above the * satisfies the inequality y > 2 - x, i.e. y > -2
- every y value directly below the * satisfies the inequality y < 2 - x, i.e. y < -2

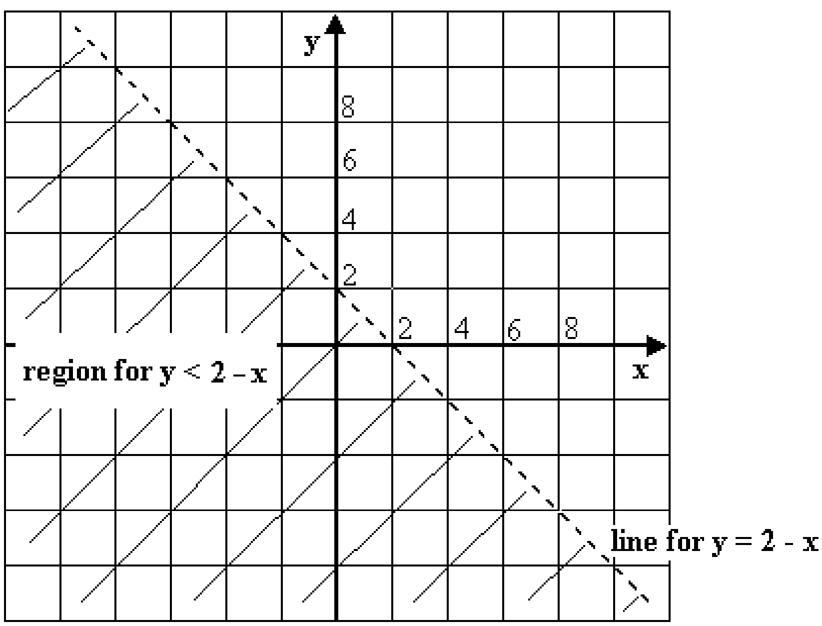
So:

- the line satisfies the equation y = 2 - x, and
- the region below the line satisfies the inequality y < 2 - x



The shaded region and the line represent the solution of $y \leqslant 2 - x$
Because the inequality includes $y = 2 - x$, we draw the line for $y = 2 - x$ solid.

The following represents the solution of $y < 2 - x$
Because this does not include $y = 2 - x$, we draw the line for $y = 2 - x$ dotted:



Part 2: ALGEBRA - contents

Graphs (Chapters 10 to 12)

Chapter 12

- TRAVEL GRAPHS

TRAVEL GRAPHS - contents

Speed and velocity

Definition: $\boxed{\text{speed} = \dfrac{\text{distance}}{\text{time}}}$

A unit of speed comes from whatever units of distance and time are used in the equation.

Example: If a car moves 100km in 2 hours, its average speed is 100km/2h = 50km/h.

Example: An athlete does the 100m sprint in 10s.

Average speed = distance/time = 100m/10s = 10m/s.

We say ‘average’ speed because the athlete starts from rest and accelerates, so his speed is changing as he runs – but he has an average speed over the 100m.

- The *standard units* used in science are called *SI units* . The SI unit for speed is m/s, which we also write as ms^{-1}

We often use the terms 'speed' and 'velocity' as if they are exactly the same. But, strictly speaking, they are not:

Definition: $\boxed{\text{velocity = speed in a particular direction}}$

So, if an object is falling at 15m/s:

- its *speed* is 15m/s
- its *velocity* is 15m/s <u>straight downwards</u>

So, to specify a velocity, we should give a direction as well as speed. However, the distinction between speed and velocity is only actually important if the direction of motion is important.

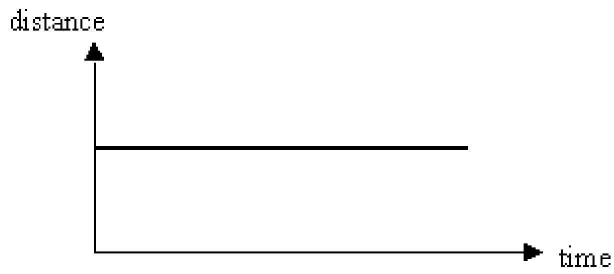
Scalars and vectors

Physical quantities are classified into two groups:

- A *scalar quantity* has size or magnitude only (i.e. just a number and a unit) - so speed is a scalar quantity
- A *vector quantity* has size *and direction* - so velocity is a vector quantity

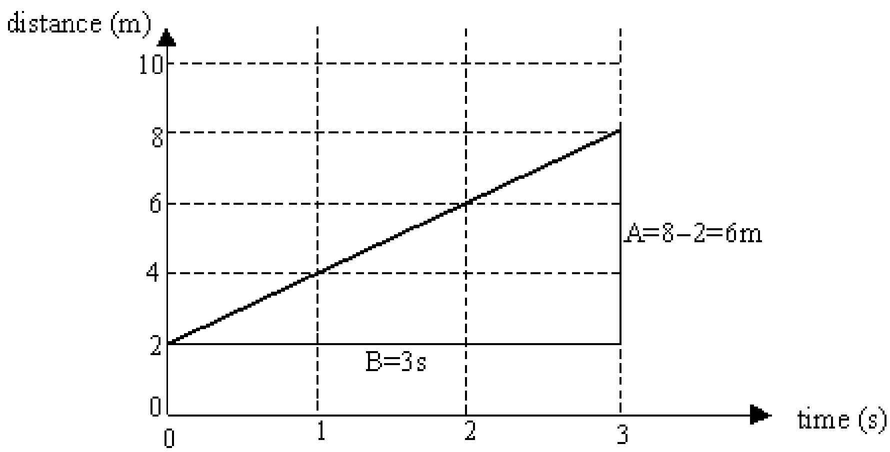
Distance-time graphs

a) Zero velocity



A horizontal line means that distance is unchanging, i.e. the object is at rest, i.e. the speed or velocity is zero.

b) Constant velocity



A straight, inclined line means that the object is moving with constant velocity.

The slope (or gradient) of the line equals the velocity. To find the slope:

- draw a triangle such as shown above
- work out A and B *from the axes* (don't measure them in cm)
- slope = A/B

In the above case: velocity = slope = 6m/3s = 2m/s

Velocity-time graphs (or v-t graphs)

There are two main quantities that can be found from a v-t graph:

1. Area below v-t graphs

- *The area below any section of a v-t graph equals the distance travelled in that section*

2. Acceleration

Ordinarily we say that a car is accelerating when it is getting faster. The more rapidly its speed/velocity increases, the greater its acceleration. This idea leads to the definition that:

- *acceleration is the rate of change of velocity*

Or,

$$\boxed{\text{acceleration} = \frac{\text{change in velocity}}{\text{time taken}}}$$

In symbols:

$$\boxed{a = \frac{v - u}{t}}$$

Where,

a = acceleration
v = final velocity
u = initial velocity
t = time

Any velocity unit divided by any time unit is a possible unit for acceleration, so we could have the unit 'miles per hour per second' or 'km/h per second' etc. However, the standard unit, i.e. the SI unit, is 'm/s per second'. This can be expressed as (m/s)/s or m/s ^{2} (said as 'metres per second squared').

A familiar acceleration is the acceleration produced by gravity. We know that if we drop something it usually falls towards the ground, getting faster as it falls. It is therefore accelerating towards the ground. Notice that since the acceleration has a direction, *acceleration is a vector quantity* .

Example

A train is moving at 60 km/h. Its speed 5 seconds later is 70 km/h.

a) What is its acceleration in km/h per second?
b) What is its speed after another 3 seconds?

a)

$$a = \frac{v - u}{t} = \frac{70 - 60}{5} = \frac{10 \text{ km/h}}{5 \text{ sec}} = 2 \text{ km/h per sec}$$

b) We can do this without an equation. An acceleration of 2 km/h per sec means that the speed is increasing by 2 km/h each second. So in 3 sec it will have increased by 6 km/h. So it is now 70 + 6 = 76 km/h.

Using the equation should clearly give the same result.

We have: $a = \frac{v - u}{t}$

We want to work out v, the final velocity, so we need to rearrange the equation.

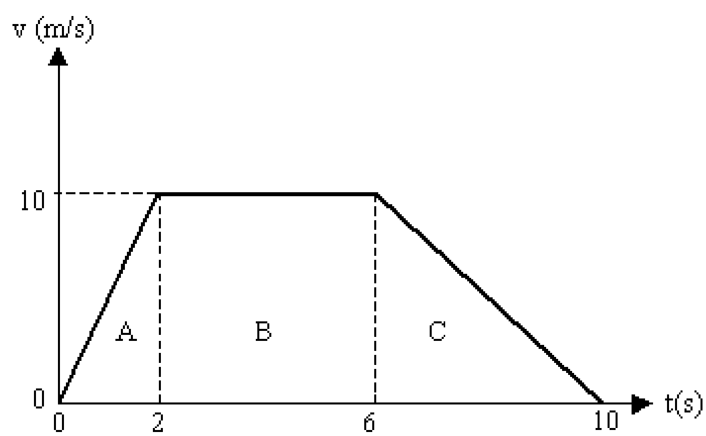
Firstly, $a\,t = v - u$ ('a t' means 'a * t')
or, $v - u = at$
or, $v = at + u$

This is usually written as:

$$\boxed{v = u + a\,t}$$

Thus, in the above problem: v = 70 + (2 * 3) = 70 + 6 = <u>76 km/h.</u>

Example



The above is a velocity-time graph for a certain object. It is made up of three sections: A, B, C.

- determine the acceleration in each section
- determine the distance travelled in each section and the total distance travelled
- describe the motion of the object

1) For each section:

$$\boxed{\text{acceleration} = \frac{\text{change in velocity}}{\text{time for change}} = \frac{\text{final velocity - initial velocity}}{\text{time for change}} = \text{slope of v-t graph}}$$

- Section A:- acceleration = (10 – 0)/2 = 10/2 = 5 (m/s)/s = 5 m/s ^{2}
- Section B:- acceleration = (10 – 10)/4 = 0/4 = 0 (zero over any non-zero number is zero)
- Section C:- acceleration = (0 – 10)/ 4 = –10/4 = – 2.5 m/s ^{2}

Notice that the acceleration is positive when the object is speeding up, and negative when it is slowing down.

2)

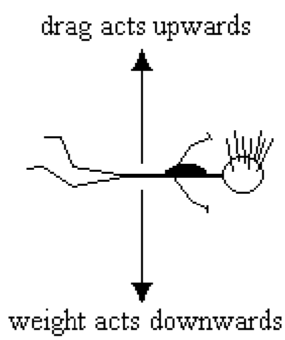
- Section A: distance = area of triangle = ½ base x height = ½ x 2 x 10 = 10 m
- Section B: distance = area of rectangle = 4 x 10 = 40 m
- Section C: distance = ½ base x height = ½ x 4 x 10 = 20m
- Total distance = 10 + 40 + 20 = 70 m

3)

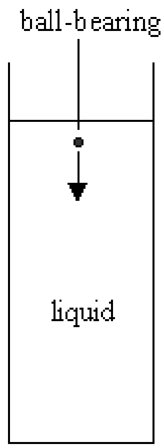
- The object starts from rest
- It accelerates for 2 sec at 5m/s ^{2} reaching 10 m/s, moving 10m
- It continues at 10 m/s for 4 sec, moving a further 40m
- Then it decelerates at a rate of 2.5 m/s ^{2} coming to rest in 4 sec, moving a further 20m.
- It moves a total distance of 70m

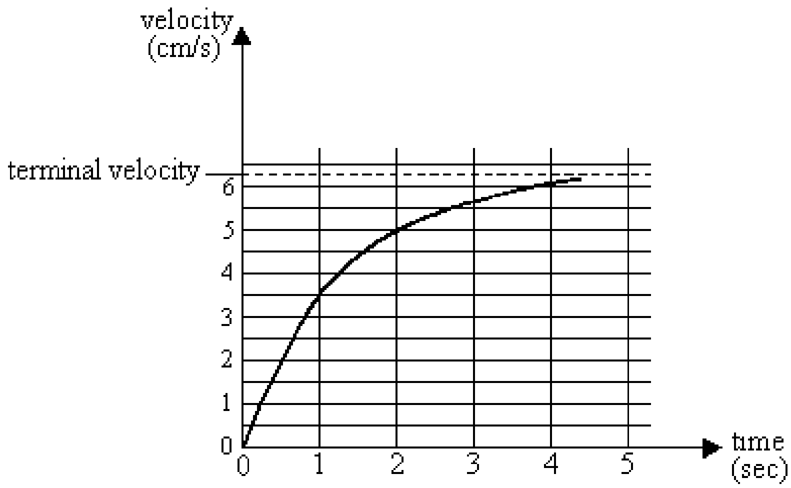
Example

When a parachutist jumps from an aeroplane s/he initially accelerates. However, the air resistance (called 'drag') acting on the parachutist increases with speed. Eventually a speed is reached at which the downward force of gravity is balanced by the drag of the air. The speed of the parachutist then stays constant at the so-called 'terminal velocity' (over 100 mph) - until the parachute is released.



We can set up something similar in the laboratory - by dropping a small ball bearing though a viscous (sticky) liquid. The following represents an experimental set-up, together with a graph of velocity against time for the falling ball bearing released from rest:



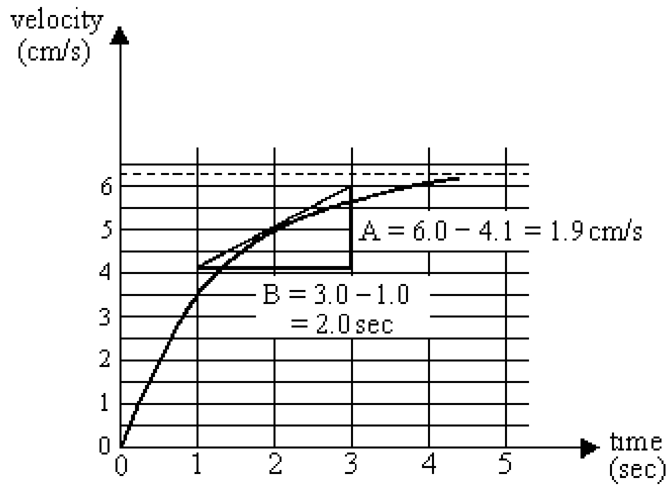


The acceleration of the ball bearing equals the gradient of the graph at any particular point.

- Determine the acceleration of the ball bearing at time = 2 seconds.

At the point on the curve corresponding to time = 2 seconds we:

- draw a tangent, and then
- add sides to turn it into a triangle, and then
- determine the gradient of the tangent - this equals the required acceleration



$$\text{acceleration} = \text{gradient} = \frac{A}{B} = \frac{1.9 \text{ cm/s}}{2.0 \text{ s}}$$
$$= \underline{0.95 \text{ cm/s}^2}$$

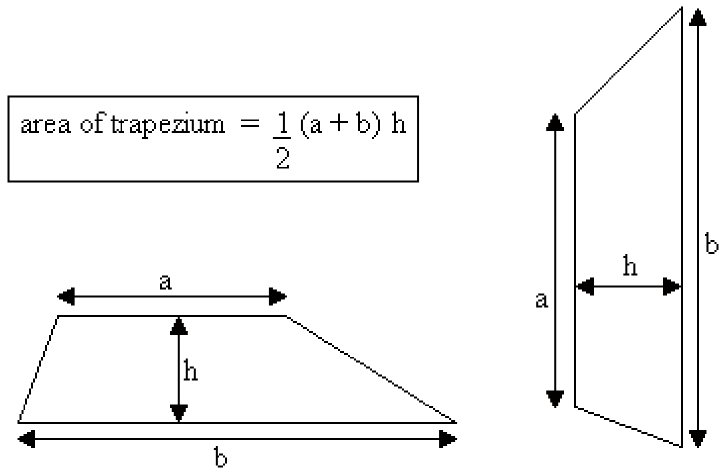
Note - $\frac{\text{cm/s}}{\text{s}} = \frac{\text{cm s}^{-1}}{\text{s}} = \text{cm s}^{-1}\text{s}^{-1} = \text{cm s}^{-2} = \text{cm/s}^2$

Trapezium rule

A trapezium is a quadrilateral (a 4-sided figure) with one pair of parallel sides, and:

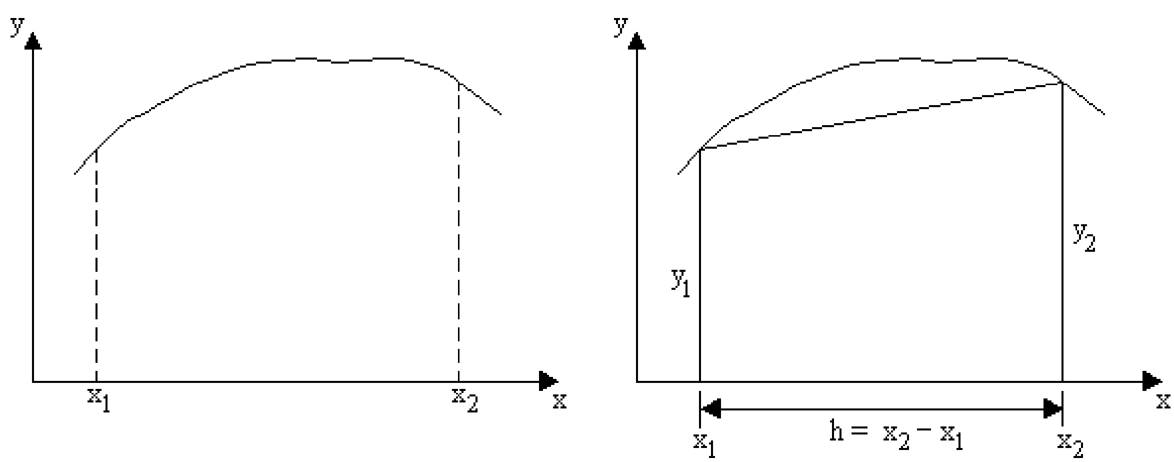
- $\text{area of a trapezium} = \frac{1}{2} * \left(\begin{array}{c}\text{sum of}\\ \text{parallel sides}\end{array}\right) * \left(\begin{array}{c}\text{perpendicular distance}\\ \text{between the parallel sides}\end{array}\right)$

The equation is the same, whether the trapezium is short and fat or tall and thin:



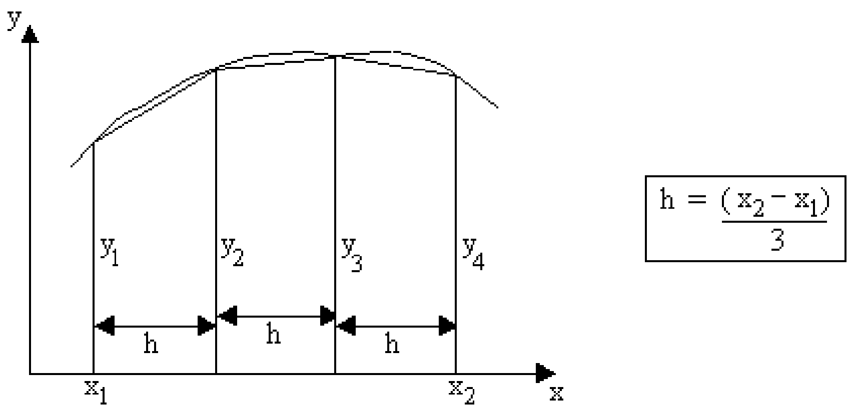
- The *trapezium rule* uses one or more trapeziums to estimate the area below a curve

In the left-hand graph below, we want to find the area between the curve and the x-axis between x_1 and x_2 . By drawing lines from x_1 and x_2 to meet the graph, and then joining their tops, we produce a trapezium, as on the right:



- The area of the trapezium = ½ * (y_1 + y_2) * h

The area of the trapezium is clearly less than the area below the curve, since a section above the trapezium is not included. However, we can improve the accuracy by using more trapeziums:



- The area of the all three trapeziums = ½*(y_1 + y_2)*h + ½*(y_2 + y_3)*h + ½*(y_3 + y_4)* h

We make the strips all the same width (h), since this lets us simplify the expression - firstly by taking h outside a bracket, since it occurs in every term:

area = [½*(y_1 + y_2) + ½*(y_2 + y_3) + ½*(y_3 + y_4)]*h
    = [½*y_1 + ½*y_2 + ½*y_2 + ½*y_3 + ½*y_3 + ½*y_4]*h

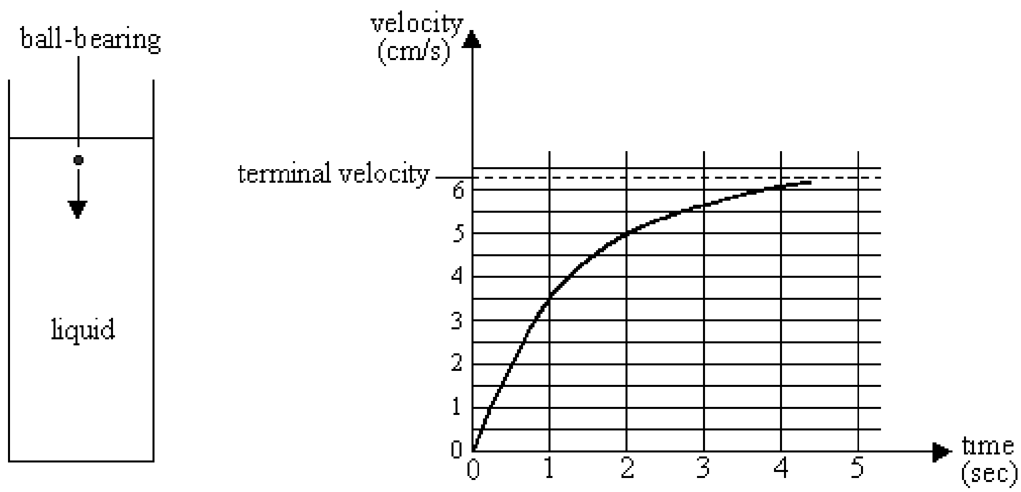
Simplifying further, we get:

$$\text{area} = \Big[\underbrace{\tfrac{1}{2}(y_1 + y_4)}_{\substack{\text{first and last} \\ \text{y values added}}} + \underbrace{(y_2 + y_3)}_{\substack{\text{in between} \\ \text{y values added}}}\Big]h$$

This same pattern is followed for any number of trapezium strips used - and the more used, the closer the area calculated is to the true area below the curve.

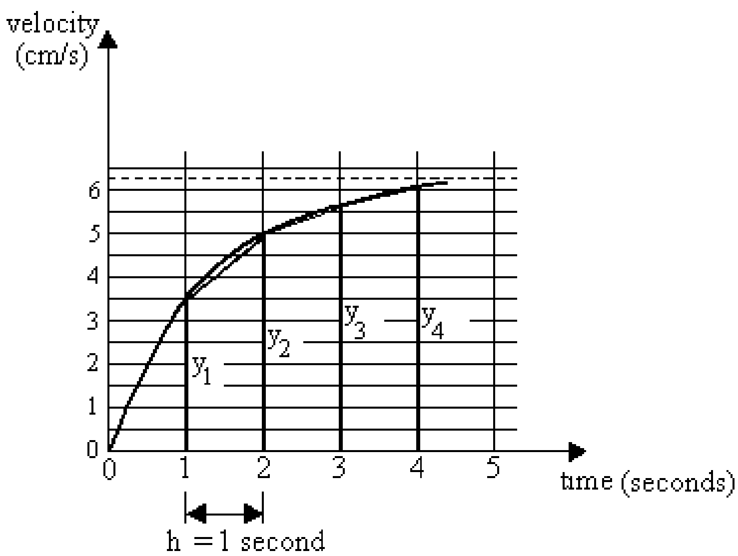
Example

The following is the v-t graph from earlier for a ball bearing falling from rest though a liquid:



Use the trapezium rule to estimate the distance fallen between 1 second and 4 seconds using three strips of equal widths.

In the following, the vertical sides labelled y_1 etc. are actually velocities, so they could just as well be labelled v_1 etc.:



$$\text{area} = \left[\frac{1}{2}(y_1 + y_4) + (y_2 + y_3)\right]h$$
$$= \left[\frac{1}{2} * (3.5 + 6.1) + (5.0 + 5.7)\right] * 1$$
$$= 4.8 + 10.7$$
$$\underline{= 15.5 \text{ cm}}$$

Notice that the unit of area = unit of a vertical side * unit of a horizontal side = (cm/s)*s = cm = unit of distance.

So, the distance travelled between 1 sec and 4 sec is approximately 15.5 cm.

Part 3: SHAPE, SPACE & MEASURES - contents

Shape (Chapters 13 to 15)

Chapter 13

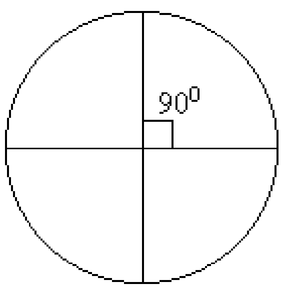
- ANGLES AND LINES
- POLYGONS

ANGLES AND LINES - contents

Angles

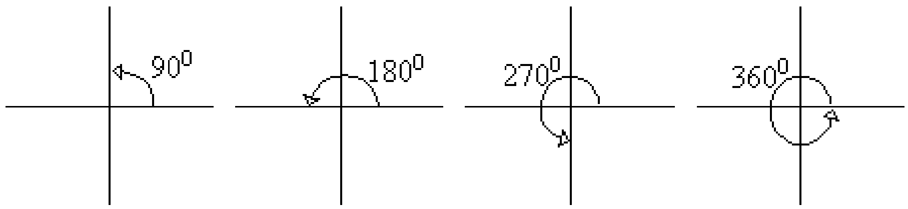
A line drawn across a circle, thought its centre, is called a diameter.

Two diameters, as below, divide a circle into 4 equals sections. If you draw this carefully on paper, and then cut out the 4 sections, they will fit exactly on top of each other.

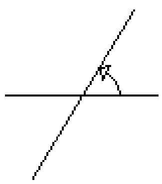
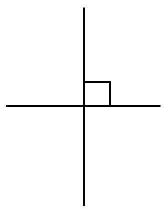
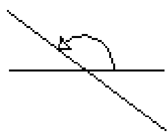
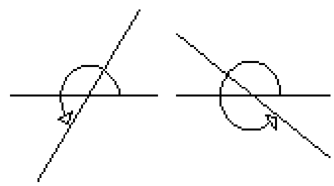


The diameter lines are said to be at *right angles* or *perpendicular* to each other.

- A right angle is defined as equal to ninety degrees (90^o)

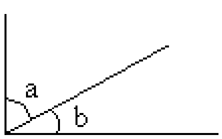


Angles that fall into certain categories have names:

less than 90^0 an *acute angle*	equal to 90^0 a *right angle*	between 90^0 and 180^0 an *obtuse angle*	between 180^0 and 360^0 a *reflex angle*
			

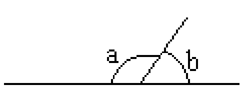
Straight lines meeting or crossing

- *complementary angles* are 2 angles which add together to make 90^o



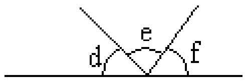
$\boxed{a + b = 90^o}$

- *supplementary angles* are 2 angles which add together to make 180^o



$\boxed{a + b = 180^0}$

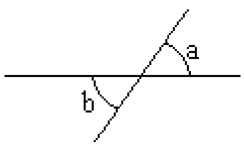
- Any number of angles that form a straight line add up to 180^o



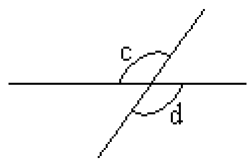
$\boxed{d + e + f = 180^0}$

- *vertically opposite angles* are formed when two straight lines cross, and:

$\boxed{\text{vertically opposite angles are equal}}$



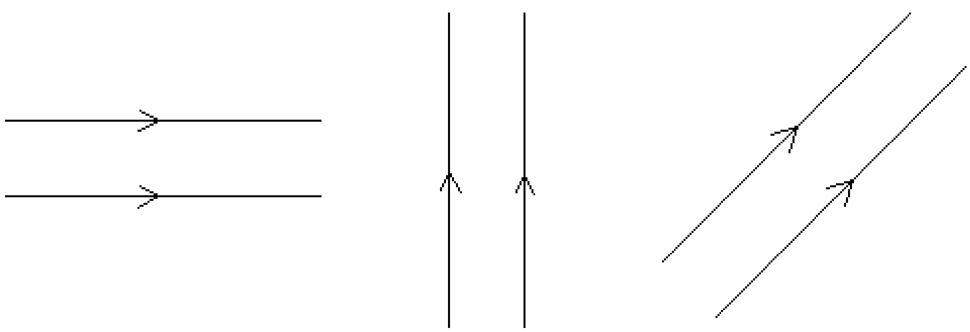
$\boxed{a = b}$



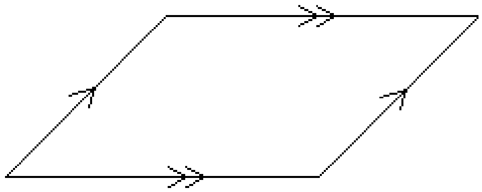
$\boxed{c = d}$

Parallel lines

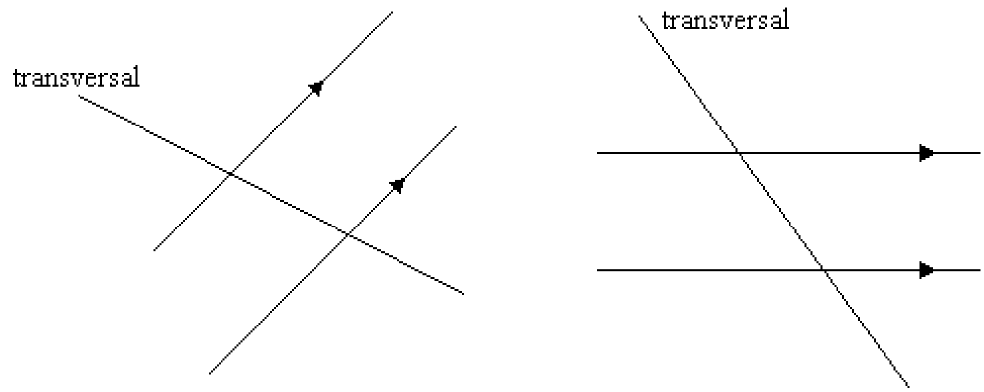
Parallel lines never meet:



In a figure, lines which are parallel are usually marked with the same number of arrows:

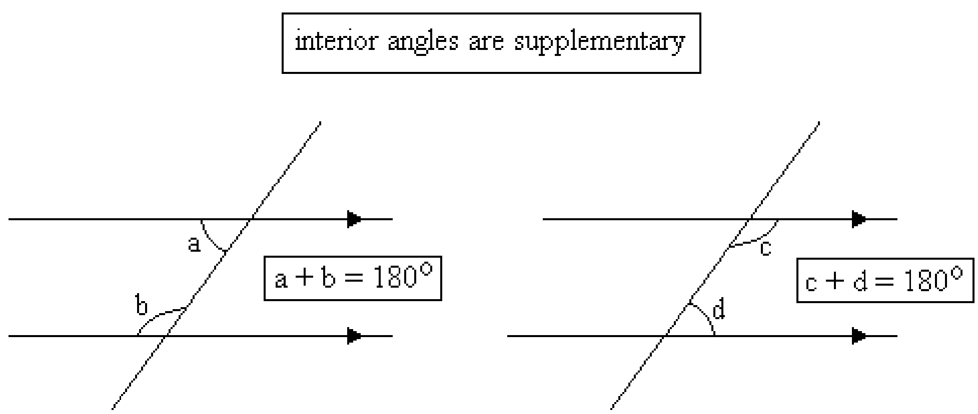


- A line drawn across 2 or more parallel lines is called a *transversal* :

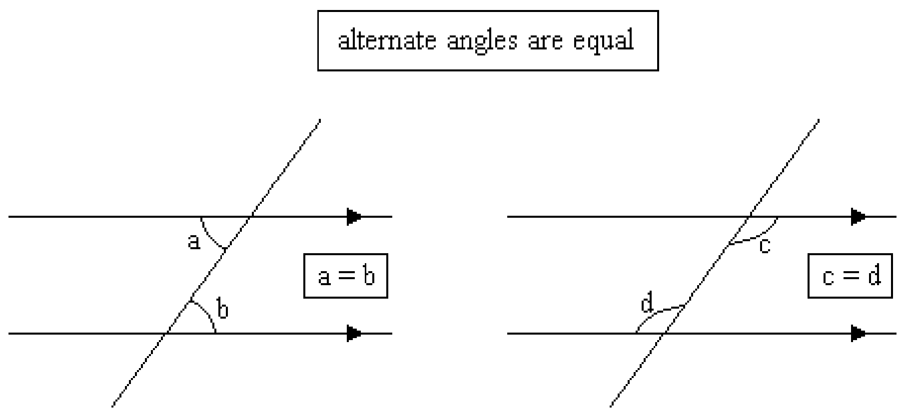


The transversal creates angles with the parallel lines, some of which obey definite rules:

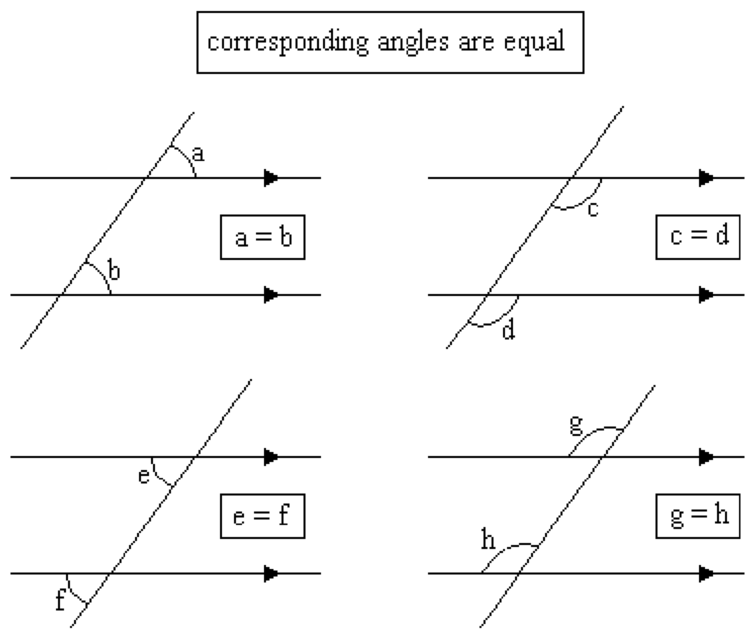
- *Interior angles* are on the same sides of the transversal, and are in between the parallel lines:



- *Alternate angles* are on opposite sides of the transversal, and are in between the parallel lines:

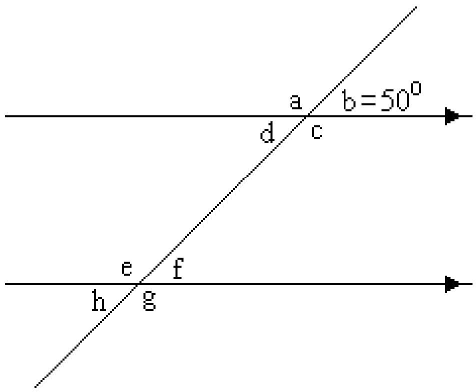


- *Corresponding angles* are on the same side of the transversal, and on the same sides of the parallel lines:



Example

Find all the angles in the following, explaining your answers:



For straight lines meeting and/or crossing:

- angles forming a straight line add to 180^o
- vertically opposite angles are equal

So,

- $a + 50^o = 180^o$, as they are on a straight line, so $a = 180 - 50 = 130^o$
- d is vertically opposite b, so $d = b = 50^o$
- c is vertically opposite a, so $c = a = 130^o$

For parallel lines:

- alternate angles are equal
- corresponding angles are equal

So,

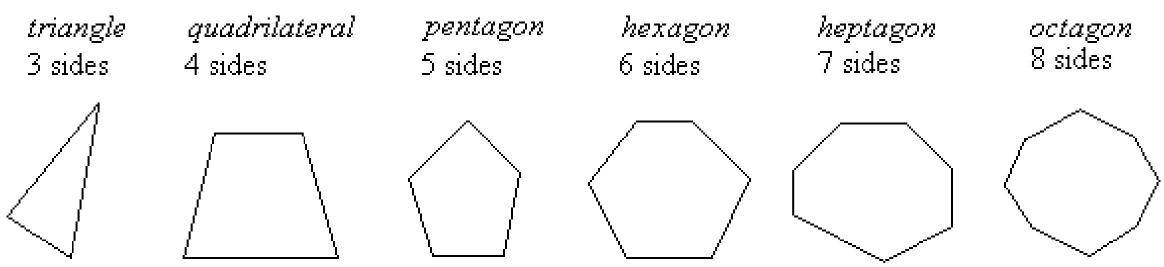
- d and f are alternate angles, so $f = d = 50^o$

- c and e are alternate angles, so e = c = 130°
- d and h are corresponding angles, so h = d = 50°
- c and g are corresponding angles, so g = c = 130°

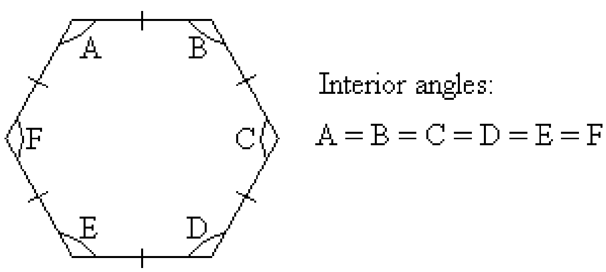
The above is just one way of working out the answers.

POLYGONS - start of this chapter - contents

A polygon is a plane (= flat) closed figure which has three or more straight sides.

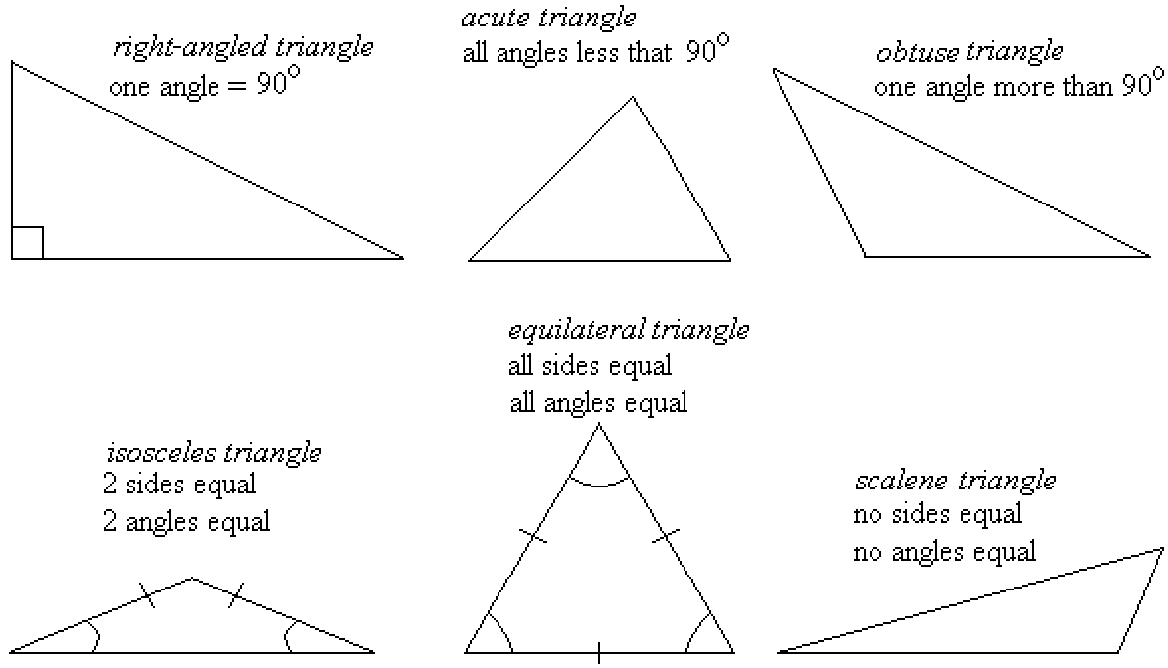


A *regular polygon* is one in which the sides are all the same length and the interior angles are all the same. For example, a regular hexagon:

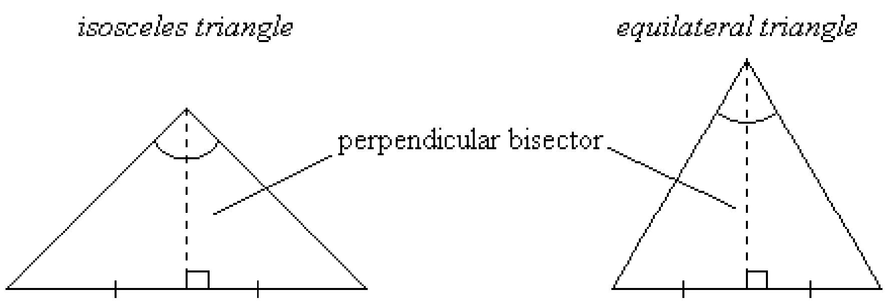


Triangles

These are the simplest polygons, each having 3 sides and 3 angles. They can be grouped into different types:



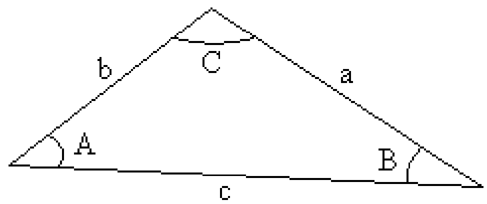
For the equilateral and isosceles triangle, if a line is drawn from the top vertex (corner), at right angles to the base, it bisects (cuts into two equal halves) the base and the vertex angle - the line is called a *perpendicular bisector* :



Sum of interior angles of a triangle

Though triangles vary in shape and size, they all have the shared property that:

- The interior angles of *any* triangle add up to 180°



$$\boxed{A + B + C = 180^o}$$

$$\boxed{\textit{the perimeter} = \text{sum of sides} = a + b + c}$$

In the equilateral triangle, the interior angles must each be 60^o (since 60^o + 60^o + 60^o =180^o).

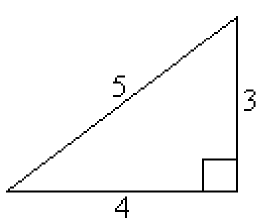
It is sometimes convenient to label any particular angle and the side opposite it with the same letter, one small and one large, as above.

Note - a perimeter is a boundary made up of lines, for example, the perimeter or edge of a garden may be rectangular:

- The length of a perimeter made up of *any* number of lines equals the sum of the lengths of *all* the lines making up the perimeter (only 3 lines for a triangular perimeter, as indicated above)

Pythagoras' theorem

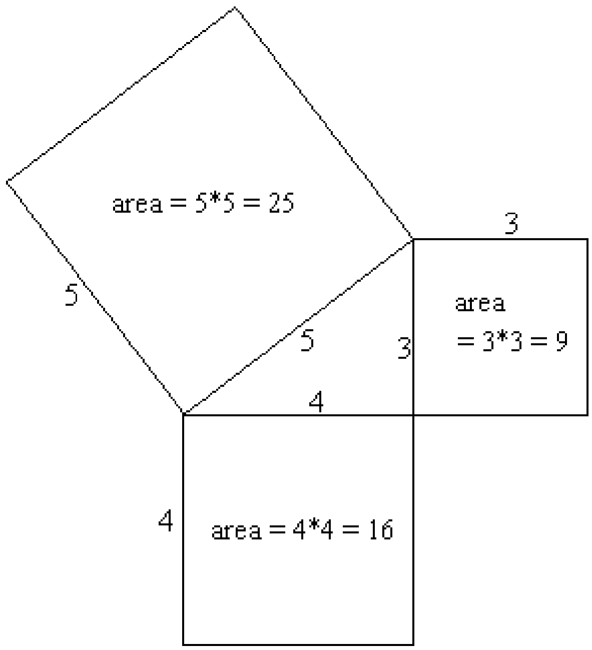
The following is a right-angled triangles with sides 3, 4 and 5 (these could all be cm, all inches or all anything else):



Note that:

- the side opposite the right-angle is called the *hypotenuse* of the triangle - it is always the *longest* side (the side marked 5 in the above case)

Suppose now that we draw a square on each side, and work out their areas:

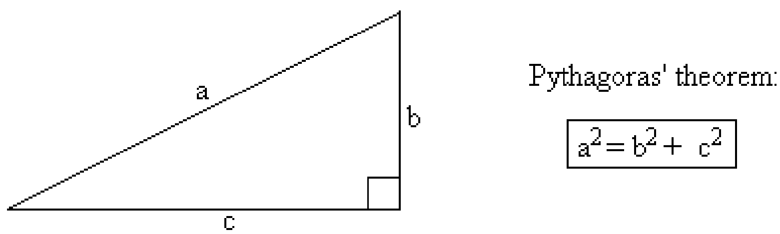


Notice that: 25 = 16 + 9, or $5^2 = 4^2 + 3^2$

This relationship is found to be true for *any* right-angled triangle, not just one with sides 3, 4 and 5. It is called *Pythagoras' theorem* :

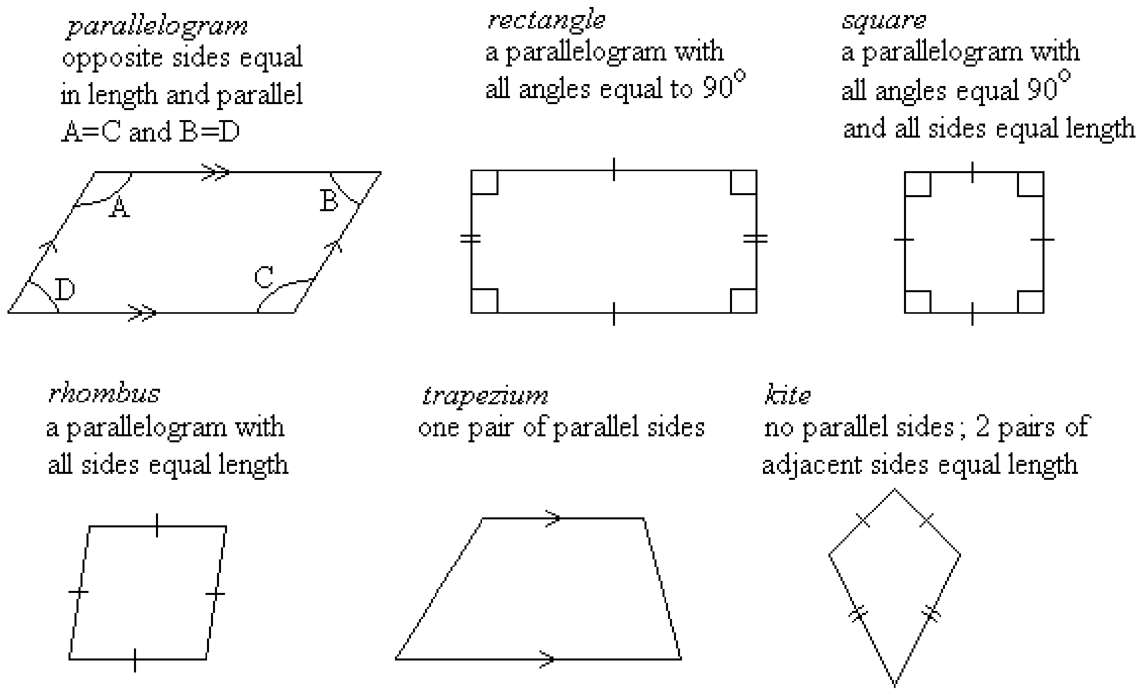
- For any right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides

We can represent this in general as:



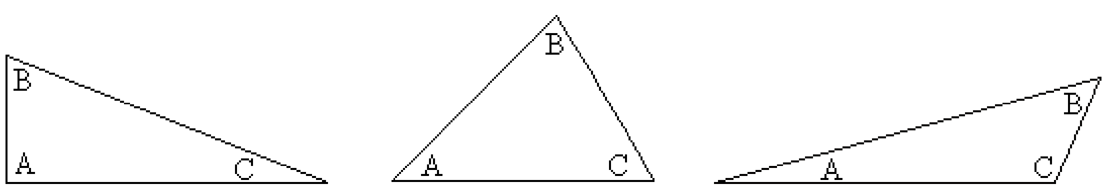
Quadrilaterals

These can be grouped into different types:

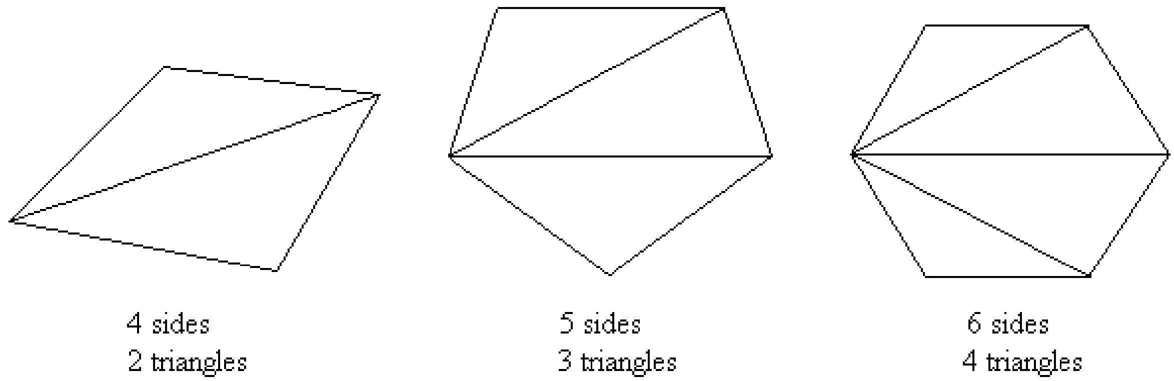


Sum of interior angles of a polygon

We've seen that in all triangles, the sum of the interior angles, A + B + C = 180^o :



We can use the above fact to find the sum of the interior angles of *any* polygon, by dividing the polygon into a set of *non-overlapping* triangles. Notice that, in the following, the triangles have been formed by drawing diagonals from one vertex (this prevents the triangles from overlapping):



Notice that:

- a quadrilateral has *4 sides* and divides into *2 triangles* , so the sum of its interior angles = $2*180^o = 360^o$
- a pentagon has *5 sides* and divides into *3 triangles* , so the sum of its interior angles = $3*180^o = 540^o$
- a hexagon has *6 sides* and divides into *4 triangles* , so the sum of its interior angles = $4*180^o = 720^o$

This pattern is followed for polygons with any number of sides - a polygon with n sides can be divided into (n - 2) non-overlapping triangles, so:

- the sum of the interior angles of a polygon with n sides = $(n - 2)*180^o$

Since equals $180^o = 2*90^o$ (= 2 right angles) the above is sometimes expressed as:

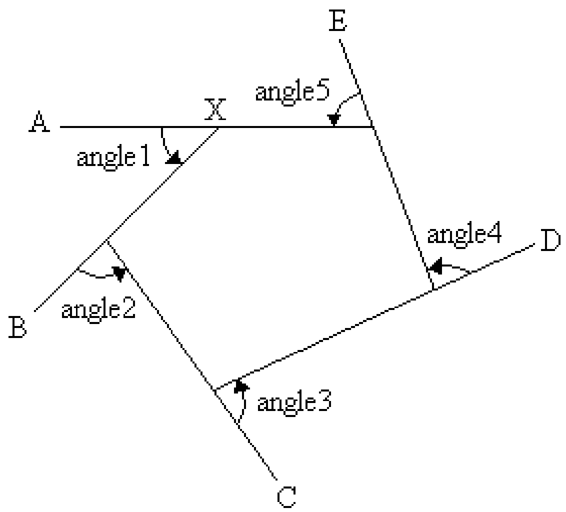
- the sum of the interior angles of a polygon with n sides = $(2n - 4)*90^o$

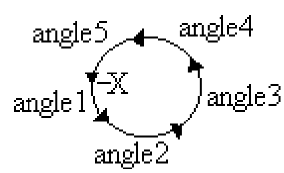
Note: For a *regular* hexagon, each *interior angle* = $720^o /6 = 120^o$. We can do a similar calculation for any *regular* polygon.

Sum of exterior angles of a polygon

The 'exterior angles' of a polygon are the ones outside it.

The exterior angles can be formed by extending each side of the polygon. In the following, starting at point X, the sides have been extended (or 'produced') in turn, going anticlockwise around the polygon:





The exterior angles are angle1, angle2, angle3, angle4 and angle5.

Notice that at every point, such as X, the exterior angle + the interior angle = 180^o , since these angles are on a straight line.

Imagine that you are standing on X, facing A, and then you:

- turn on the spot though angle1
- then turn through angle2
- then turn through angle3
- then turn through angle4
- finally turn through angle5

You have turned through a complete circle, 360^o . This may be clearer if you look at the right hand diagram, where the angles have been joined together.

This is the case, for a polygon of *any* number of sides, so:

- The sum of the exterior angles of any polygon = 360^o

GCSE Maths Notes

Part 3: SHAPE, SPACE & MEASURES - contents

Shape (Chapters 13 to 15)

Chapter 14

- CONGRUENCE AND SIMILARITY
- SYMMETRY
- CIRCLES
- GEOMETRICAL CONSTRUCTIONS

CONGRUENCE AND SIMILARITY - contents

Congruence

Congruent shapes have the same shape and size.

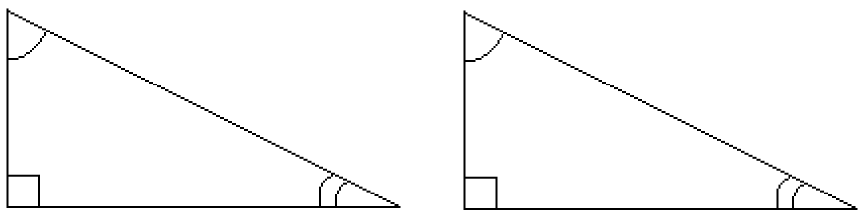
- Solids may be congruent - in a dinner service, some plates will be congruent and some dishes will be congruent, etc.

- Figures drawn on paper may be congruent - if two shapes are cut out, and they can be fitted exactly on top of each other, then they are congruent

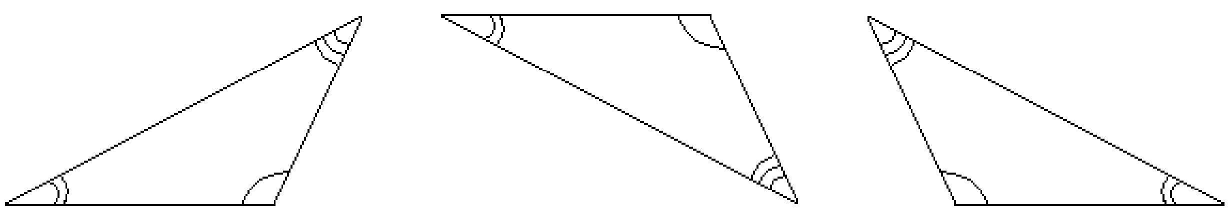
Figures are congruent if:

1. all corresponding angle are equal, and
2. all corresponding sides are equal

The following are congruent triangles:



The following are also congruent triangles:



To test for congruence between two figures you could:

- cut them out and see if they fit exactly on top of each other, or
- measure and label every side and angle on both figures, and see if they have corresponding sides and angles the same

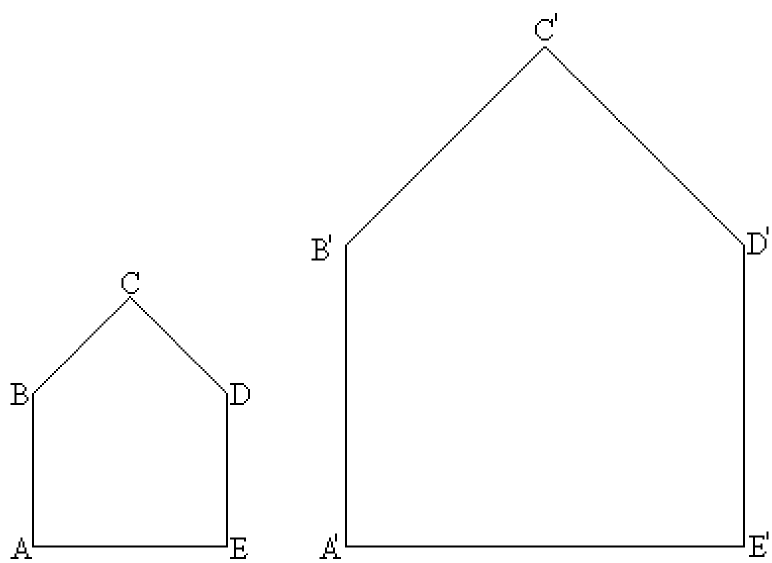
Similarity

Figures are similar if:

1. all corresponding angle are equal, and
2. all corresponding sides are in the same ratio

(1) means that similar figures have the same shape, and (2) means that they may have different sizes.
[Figures which are identical (congruent) are similar, and the ratio of corresponding sides is 1:1]

The following are similar figures - all the sides are in the ratio 1:2.



Similar triangles

As already said, two figures are similar if:

1. all corresponding angle are equal, and
2. all corresponding sides are in the same ratio

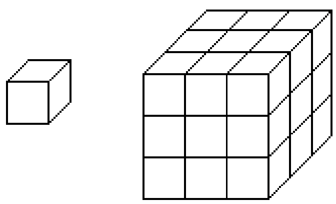
In the case of triangles, we do not need to measure all the angles and all the sides to see if two triangles are similar. Two triangles are similar if any one of the following is true:

1. all three corresponding angles are equal, or

2. the ratios of corresponding sides are equal, or
3. the ratio of two corresponding sides are the same, and the angle between them are equal

Area and volumes

3D (3 dimensional) objects can also be similar. The following are similar - all the lengths are in a ratio of 1:3.



We see also that:

- The area of a side of the large cube is 9 (= 3^2) times the area of the corresponding side of the small cube
- The volume of the large cube is 27 (= 3^3) times the volume of the small cube

The number 3 is called the 'linear scale factor' - it is the amount by which we multiply each length to change the small cube into the large cube.

In general, if two objects are similar, with a linear scale factor of S, then:

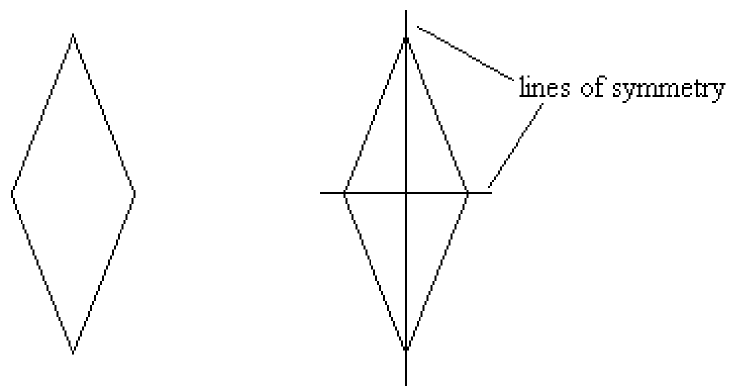
- the area scale factor is S^2
- the volume scale factor is S^3

SYMMETRY - start of this chapter - contents

1. Reflexive symmetry

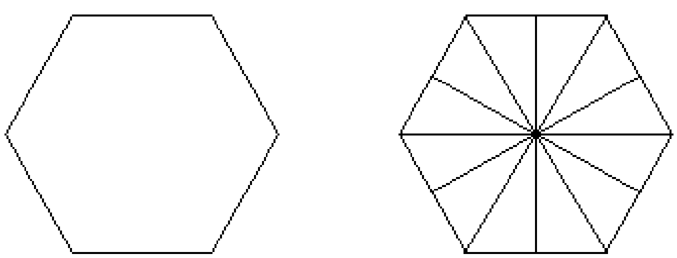
- A figure has reflexive symmetry if a line can be drawn through it, dividing it in two, and one half is the mirror image of the other half

Such a line is called a *line of reflexive symmetry* , or just a *line of symmetry* - some figures have more than one:



You can check a line of symmetry by placing the straight edge of a plane (= flat) mirror on the line and viewing the reflection.

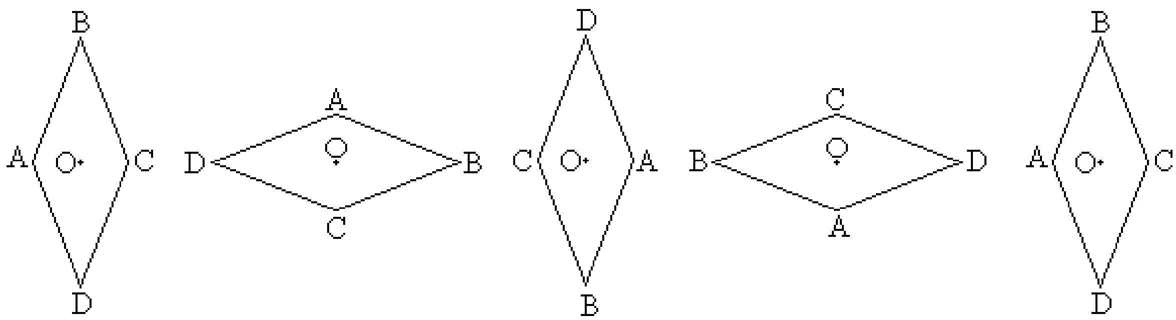
The following represents a regular hexagon and its six lines of symmetry:



2. Rotational symmetry

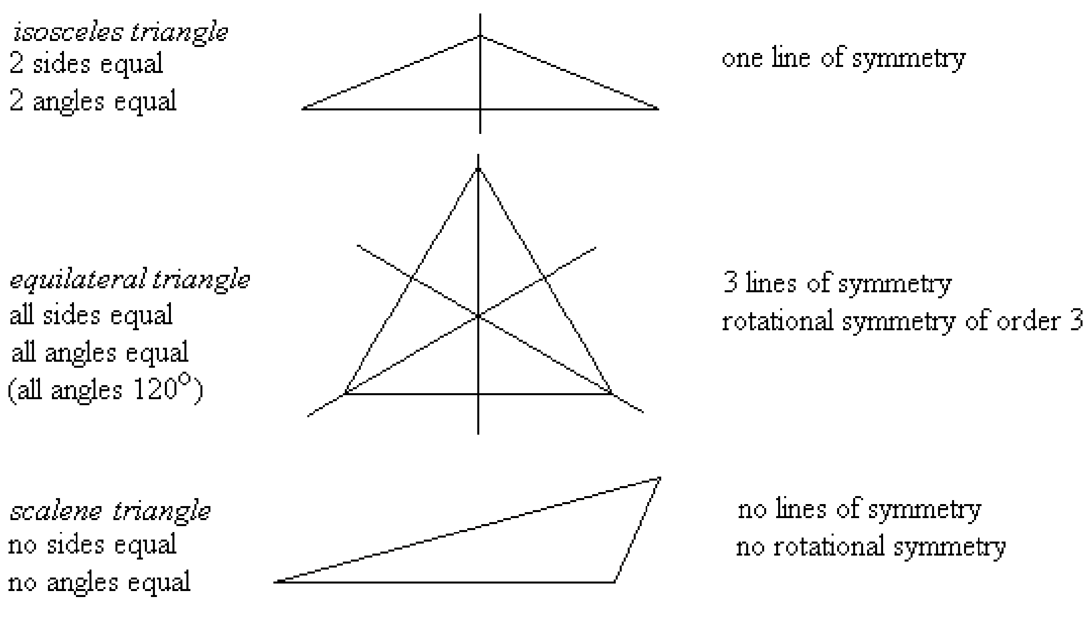
The following diamond shape has reflexive symmetry about AC and BD, as seen above.

Suppose now that it is rotated about its geometric centre, O, until it arrives back at its original position:



Since it looks the same in two positions, the shape is said to have rotational symmetry of order 2 about the *point of rotational symmetry* , O. (We only count the 1st and 3rd diagrams, since in the 5th the shape is back in its original position)

Triangles

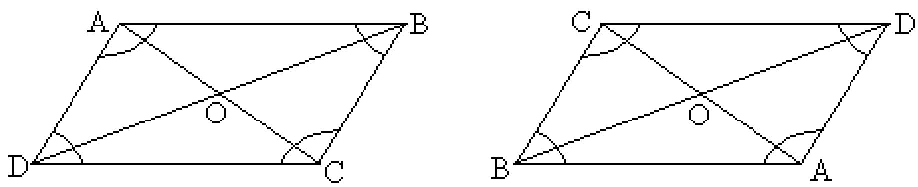


We can use symmetry to work out some properties of various shapes:

Parallelograms

1. Parallelogram - opposite sides parallel

In the following, the letters A etc., are used to label the corners, but they also represent the interior angles at the corners. If we rotate the left-hand parallelogram about O though 180° it becomes the right-hand diagram. These diagrams are exactly the same, so the parallelogram has rotational symmetry about O. It only looks the same in these two positions when rotated, so we say that it has rotational symmetry of order 2.



These diagrams are exactly the same - and we see that:

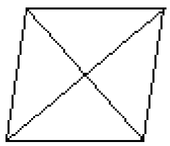
- AB is where DC was, so AB = DC (similarly AD = BC)
- A is now where C was, which means that angle A = angle C (similarly angle B = angle D)
- OA is where OC was so OA = OC (similarly OB = OD)

So, the rotational symmetry implies that:

- opposite sides are equals (AB = DC; AD = BC)
- opposite interior angles are equal ($\angle A = \angle C$; $\angle B = \angle D$) [note: $\angle A$ means 'angle A', etc.]
- diagonals bisect each other (OA = OC; OB = OD)

2. Rhombus - a parallelogram with all sides equal length

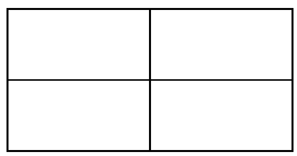
Since a rhombus is a type of parallelogram, the above properties apply to it. But also:



the diagonals are lines of symmetry - so:

- the diagonals bisect opposite angles
- the diagonals bisect each other at right-angles

3. Rectangle - a parallelogram with all angles right-angles

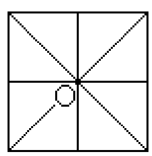


two lines of symmetry - so:

- the diagonals have equal length

4. Square - a rectangle with all sides equal

Since a square is a type of rhombus, the above properties of a rhombus apply to it. But also, it has:

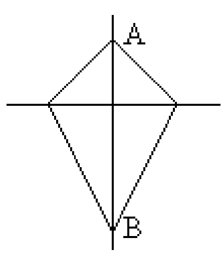


4 lines of symmetry
rotational symmetry of order 4 about O - so:

- all angles are either 90° or 45°

Kites

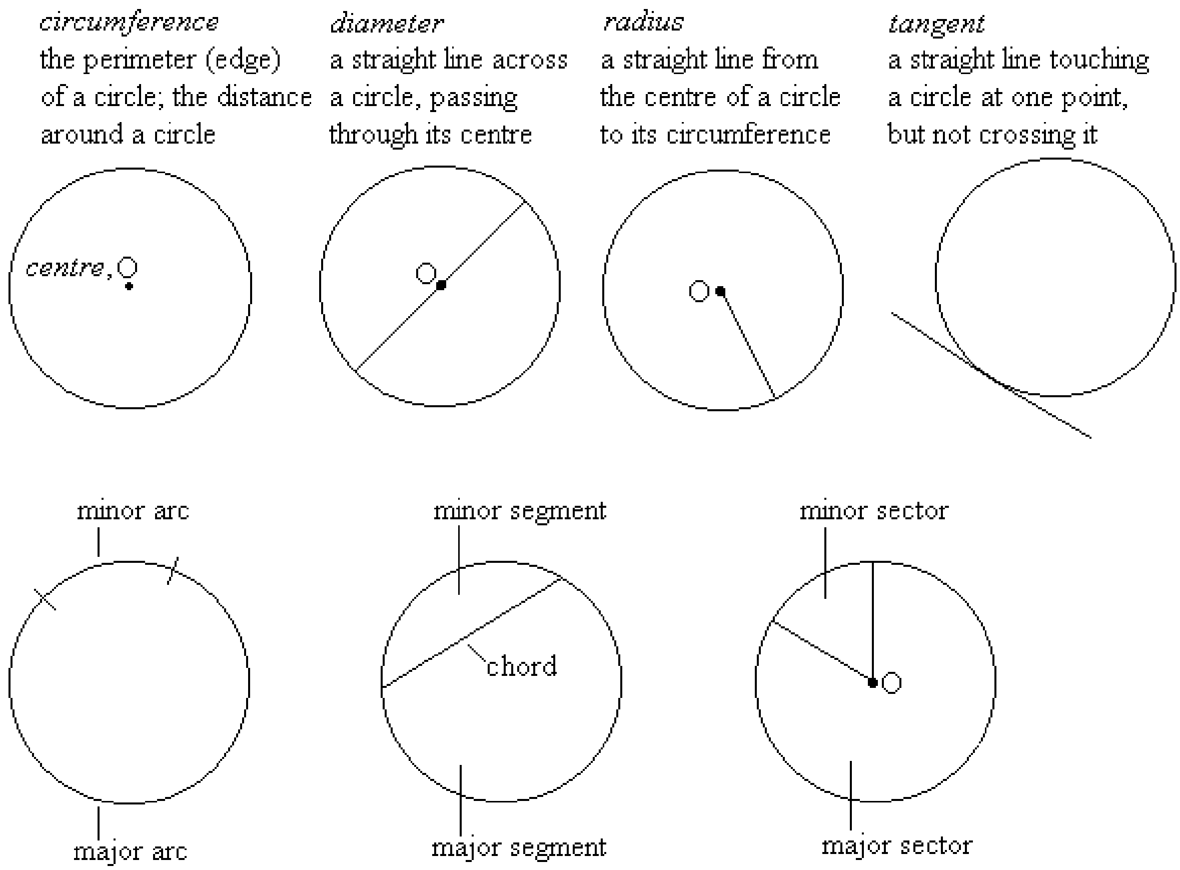
A kite has no parallel sides and has 2 pairs of adjacent side equal length - also:



- the diagonals bisect each other at 90°
- the diagonals bisect the interior angles at A and B

CIRCLES - start of this chapter - contents

Terminology

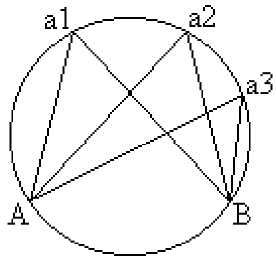


There are various rules that apply to circles, lines and angles - some of these are called 'theorems'.

Subtended angles

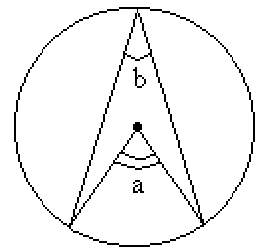
The phrase 'the angle subtended by an arc at a point' means the angle opposite to the arc, whose size is determined by the size of the arc:

angles subtended by an arc at the circumference in the *same segment* of a circle are equal



a1 = a2 = a3

the angle subtended by an arc at the centre of a circle is twice the angle the same arc subtends at the circumference



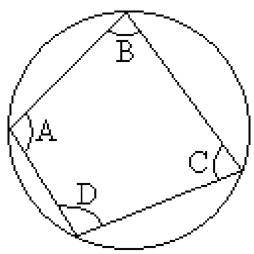
a = 2b

Cyclic quadrilaterals

- A cyclic quadrilateral is one with all its corners touching a circle

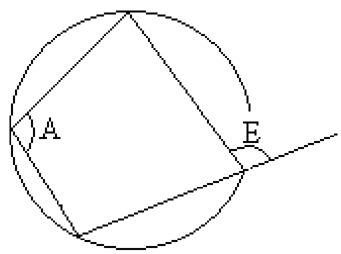
the opposite angles of a cyclic quadrilateral add up to 180°

$A + C = 180^{\circ}$ $B + D = 180^{\circ}$



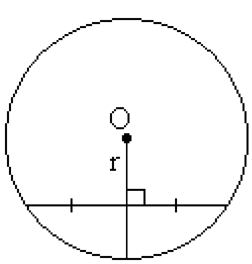
the exterior angle of a cyclic quadrilateral equals the opposite interior angle

$A = E$

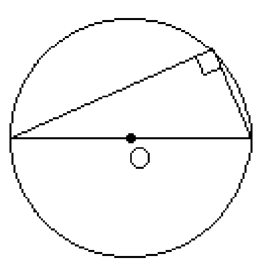


Right-angles in circles

a radius at 90° to a chord divides the chord into two halves

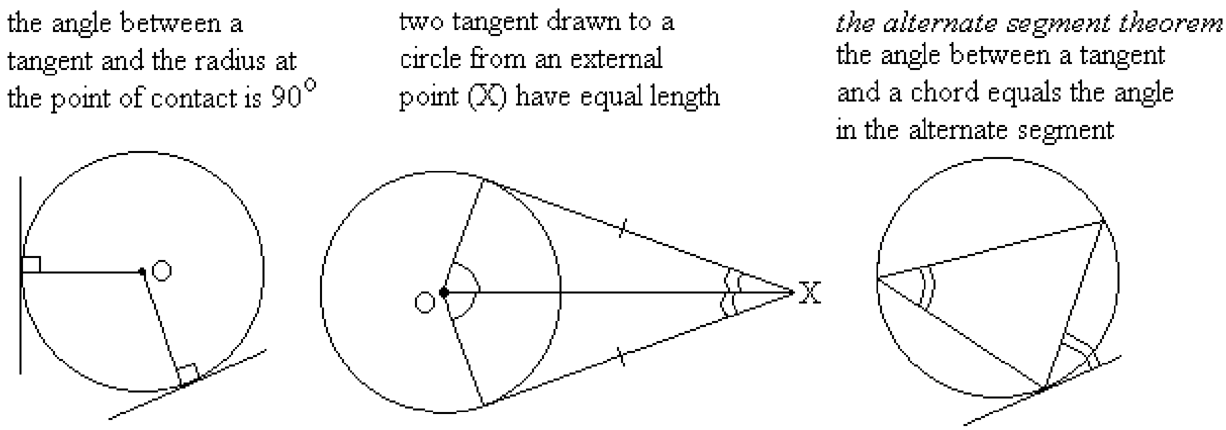


a triangle formed in a semicircle is a right-angled triangle



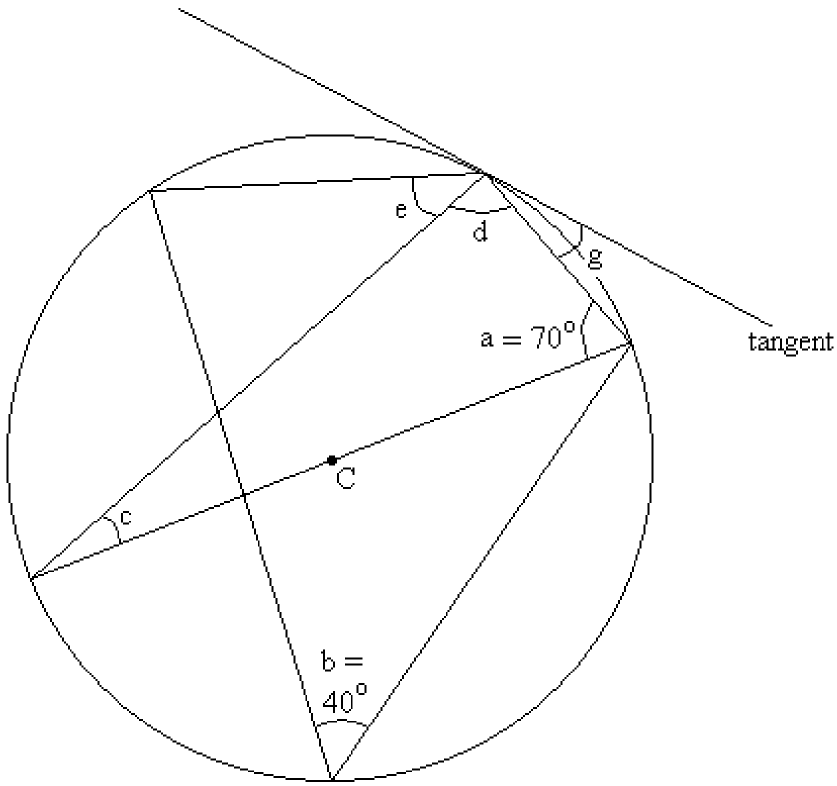
In the left-hand diagram, the radius r is referred to as the 'perpendicular bisector of the chord'.

Tangents



Example

In the following, C is the centre of the circle. Work out the unknown angles.



a, c and d are angles in a triangle which has been drawn in a semicircle, so d is right angled triangle:

- d = 90°
- so, c = 90 - 70 = 20°

By the alternate segment theorem, g = c = 20°

b and (e + d) are opposite angles in a cyclic quadrilateral, so:

- b + d + e = 180°
- so, 40 + 90 + e = 180°
- so, e = 180 - 40 - 90 = 50°

Area of a circle

Many centuries ago it was discovered that if the circumference of *any* circle were divided by its diameter, the *same number* was always obtained, 3.142 (to 3 decimal places).

This number is usually denoted by the Greek letter π ('pi'). Thus, by definition:

$$\boxed{\pi = \frac{\text{circumference}}{\text{diameter}}}$$

So, $\boxed{\text{circumference} = \pi * \text{diameter} = 2\pi r}$ (r = radius of circle)

You may have π on your calculator. If not, you can use 3.14 or 22/7 for its value.

It can be shown that the *area* of a circle can be found from:

$$\boxed{\text{area} = \pi r^2}$$

Example

Find the area of a circle of radius 5cm.

Area $= \pi r^2 = \pi * 5^2 = \underline{78.5 \text{ cm}^2}$

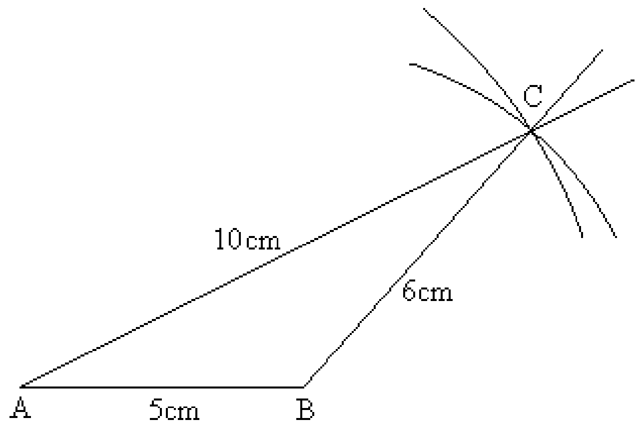
GEOMETRICAL CONSTRUCTIONS - start of this chapter - contents

Triangles

A triangle has 3 sides and 3 angles, but you don't need to know *all* of them to draw one.

Example

Draw a triangle with sides 5cm, 6cm and 10cm.



To draw the above, the procedure was to:

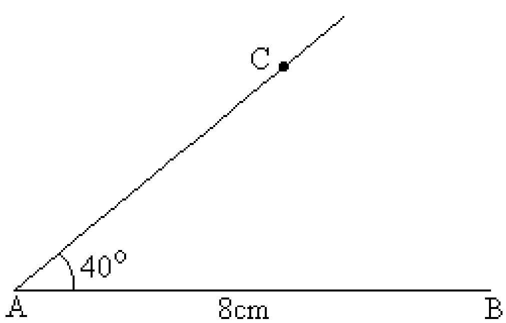
- draw a 5cm horizontal line, labelled AB
- set a pair of compasses to 10cm, put the point on A, and draw an arc (any length, but not too short)
- set the compasses to 6cm, place the point on B and draw another arc, to cross the first one
- label the point where they cross as C, and draw the lines shown

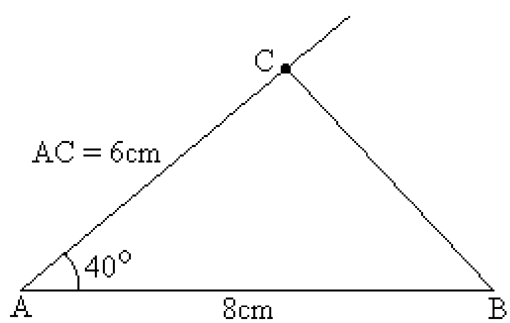
The question does not say which way to draw the triangle, so you could choose to, for example, to draw AB as 10cm, and then from the ends of it, draw 5cm and 6cm arcs.

Example

Draw a triangle with two sides 8cm and 6cm with an included angle of 40^o .

Note: the 'included angle', means the one between the two lines.



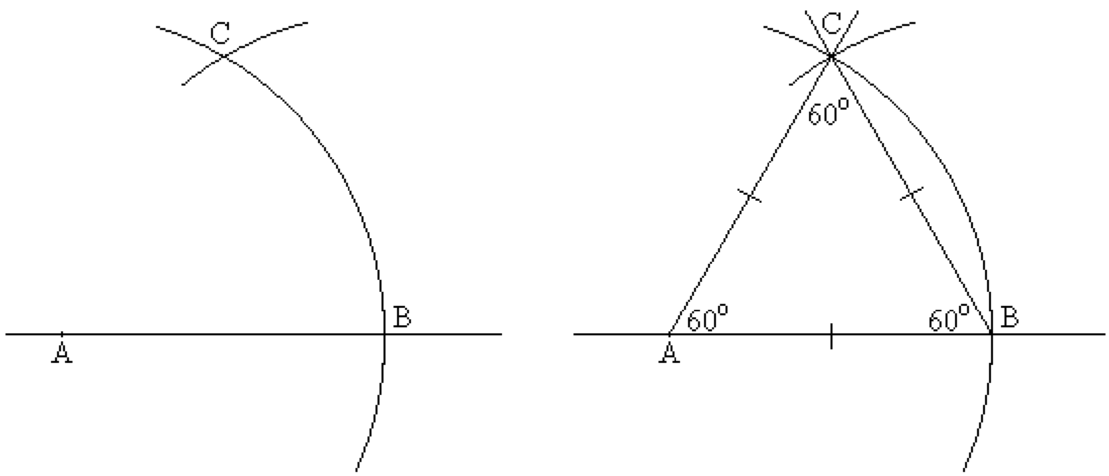


To draw the above, the procedure was to:

- draw the horizontal line AB, 8cm long
- use a protractor to draw a line at 40^o to AB
- mark the point C, 6cm along the line
- join C to B

Drawing an equilateral triangle

We draw a horizontal line, and mark a point A. Then set a pair of compasses to, say, 6 cm and draw a large arc using A as the centre. In the following the arc cuts the horizontal line at B:



Without altering the compasses, draw another arc, using B as the centre, to cut the first arc - at C in above diagram.

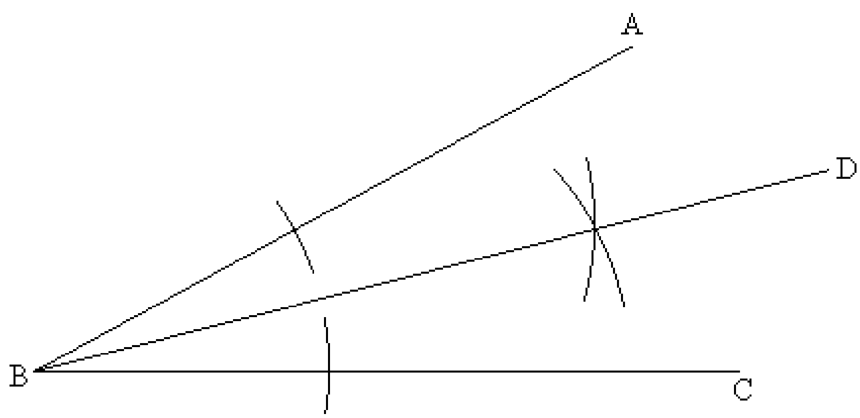
Since the compasses were not altered, then AB = AC = BC. So we just join these together to get the equilateral triangle ABC.

This is a method of producing an equilateral triangle but can also be used if we just want to draw a 60^o angle, since all the angles inside the triangle are 60^o .

Bisector of an angle

- To 'bisect' something means to divide it into equals halves

Here we want to bisect the angle ABC by the line BD. The following is the final diagram, and the procedure to produce it is after the diagram:



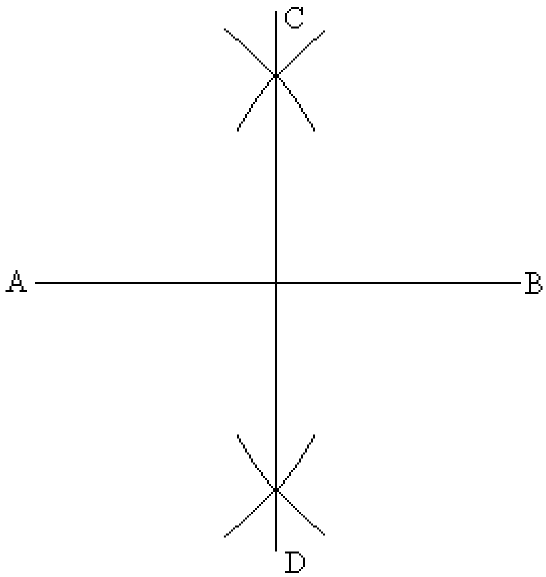
1. with B as the centre, use a pair of compasses to draw arcs of equal radius to cut AB and BC
2. using the points where the arcs cut AB and BC, draw two more arcs of equal radius, such that they cut each other
3. draw BD though the point where the arcs cut

The line BD bisects the angle ABC.

Note: we sometimes use the symbol $\angle$ABC to stand for 'angle ABC'

Perpendicular bisector of a line

Here we want to draw the 'perpendicular bisector' of line AB, i.e. a line at right angles to AB and which divides AB in half:



- With A and then B as the centre, draw arc of equal radius above and below AB, so that the arc cross as shown (the radius must be something bigger than half of AB, or else the arcs will not cross)
- Draw the line CD as shown - this is the required perpendicular bisector of AB

Part 3: SHAPE, SPACE & MEASURES - contents

Shape (Chapters 13 to 15)

Chapter 15

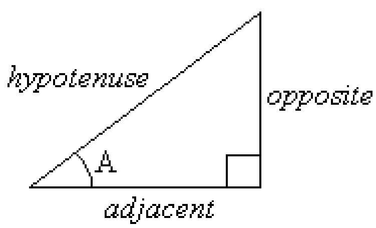
- TRIGONOMETRY

TRIGONOMETRY - contents

In a right-angled triangle, apart from the right-angle and the hypotenuse, there are two other angles and two other sides.

Relative to one of the other two angles, such as A in the diagram below:

- one of the other two sides is *opposite* to A, and
- the other side is *adjacent* to A



We define the following quantities for angle A:

- The ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is called the *sine* of angle A
- The ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ is called the *cosine* of angle A
- The ratio $\frac{\text{opposite}}{\text{adjacent}}$ is called the *tangent* of angle A

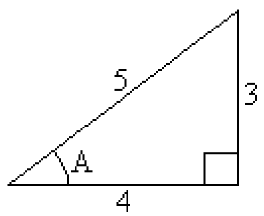
We shorten sine to *sin* , cosine to *cos* and tangent to *tan* , so the above can be expressed as:

$$\boxed{\sin A = \frac{\text{opposite}}{\text{hypotenuse}}} \quad \boxed{\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}} \quad \boxed{\tan A = \frac{\text{opposite}}{\text{adjacent}}}$$

Note: We usually use degrees for measuring angles, but there are other angle units also used, called 'rads' (short for radians) and 'grads'. You may have a button labelled DRG on your calculator, which lets you change between degrees, rads and grads. Check that your calculator is set for degrees before doing the following example:

Example

Work out sin A and then the angle A in the triangle:



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

Do the above calculation on your calculator.
To get the angle A, what you now press will depend on the type of calculator.
You will probably have to press either **inv sin** or **sin^{-1}** or **2nF sin** .
You should get the answer 36.9^{o} (to 1 d.p.)

Note - on some calculators you would have to press **sin^{-1}** and then enter (3/5) and then press 'equals'.

Example

Repeat the above, but use (i) cos A and (ii) tan A

(i) $\cos A = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{4}{5} = 0.8$

Now pressing **inv cos** (or the equivalent) on the calculator gives:

$\underline{A = 36.9^o}$ (to 1 decimal place)

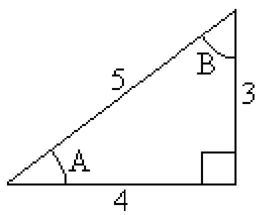
(ii) $\tan A = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{3}{4} = 0.75$

Now pressing **inv tan** (or the equivalent) on the calculator gives:

$\underline{A = 36.9^o}$ (to 1 decimal place)

Example

Find the angle B in the following:



To use sines etc. you need to identify the hypotenuse of the triangle and then the opposite and adjacent for the angle you're interested in.

The hypotenuse is opposite the right-angle, so that is the 5, as before.

- The opposite to B is now the 4 and the adjacent to B is the 3.

We could use any of sin B, cos B or tan B - let's use sin B:

$\sin B = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{4}{5} = 0.8$

Pressing **inv sin** now gives $B = 53.1^o$ (to 1 decimal place)

Check : Recall that the angles in a triangle always add up to 180^o

In the above triangle, we have A = 36.9^o , B = 53.1^o , and the right angle, which is 90^o

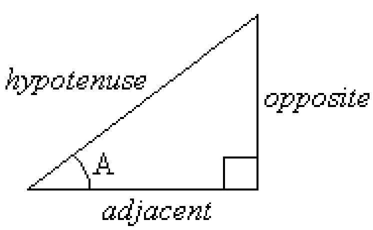
We see that $36.9^o + 53.1^o + 90^o = 180^o$, as it should be.

Note:

- sines, cosines and tangents are called *trigonometric functions* - they are all expressed as ratios of the sides of a right-angled triangle
- *trigonometry* is the branch of maths concerned with the properties and uses of trigonometric functions - trigonometry has important applications in surveying and navigation etc.

Sine, cosine and tangent for any size angle

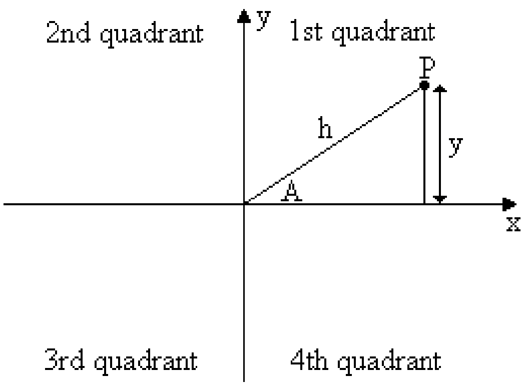
We've already seen the basic definitions of sine ('sin'), cosine ('cos') and tangent ('tan') for a right-angled triangle:



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}}$$

In the above triangle, angle A can vary between 0^o and 90^o . However, the above definitions can be extended so that we can find sine etc., for *any* angles from 0^o to 360^o .

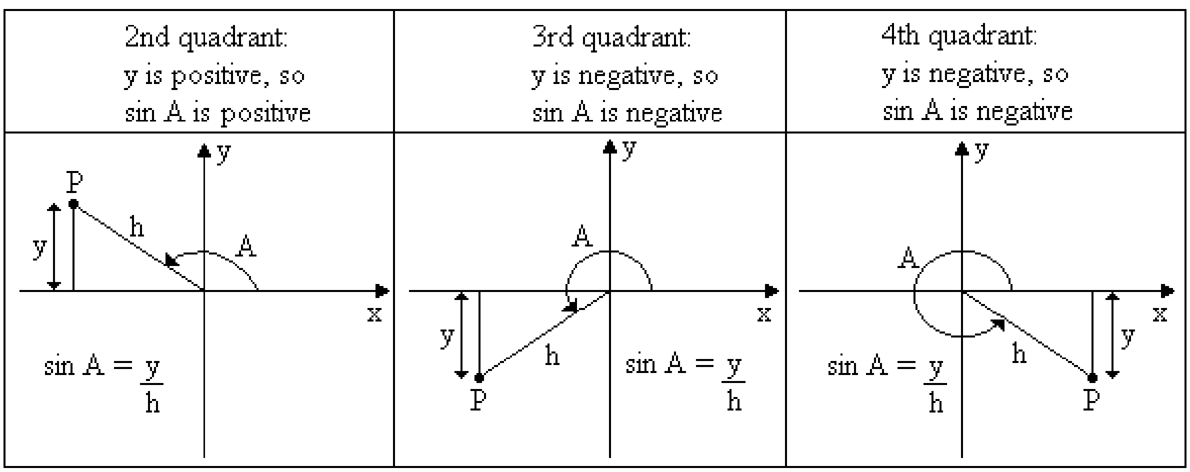
Suppose that we redraw the triangle as below. We see that the x-y axes create four 'quadrants', and we draw the triangle in the 1st quadrant. The opposite to angle A is now the y-coordinate of the point labelled P:



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{y coordinate of P}}{\text{hypotenuse}} = \frac{y}{h}$$

Sin A is positive in the 1st quadrant, i.e. for A between 0^o and 90^o , because y is positive in the first quadrant (and we always take h to be positive).

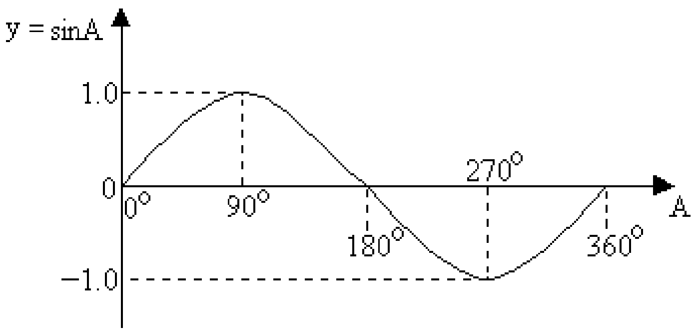
Now, if for any value of A we think of the 'opposite' side to A as being the y-coordinate of the point P, we can work out sin A for any angle. The sign (i.e. positive or negative) of sinA will depend on the sign of y (since h is always positive):



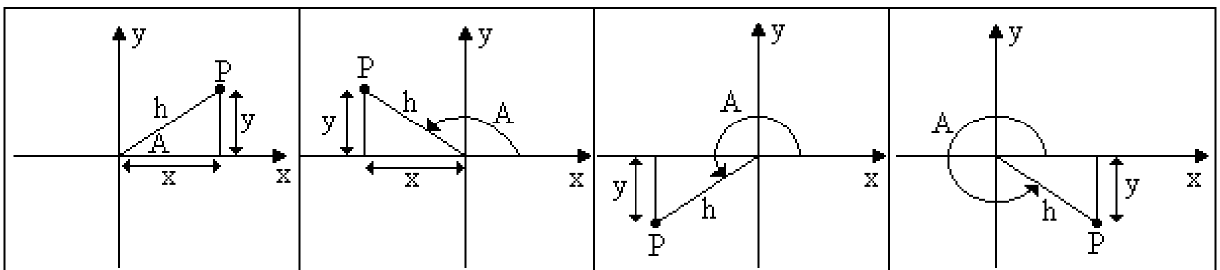
Check that your calculator is set to degrees (not rads or grads), and calculate the sines for the angles below - and notice that the values are always between +1 and -1:

A	0	30	60	90	120	150	180	210	240	270	300	330	360
y = sinA	0.00	0.50	0.87	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00	-0.87	-0.50	0.00

The way that sinA varies can be seen better if the above points are used to plot a graph:



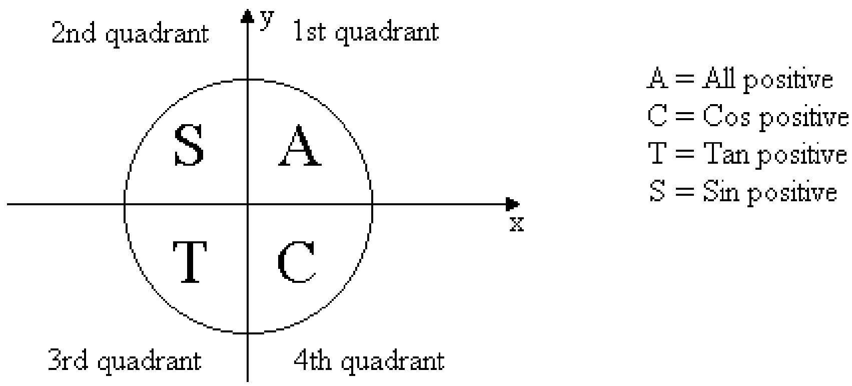
By using the length h and the coordinates of point P to work out sin etc. we can work out the values of sin, cos and tan for any angles, as indicated below:



1st quadrant: x is positive (+) y is positive (+) sin = y/h = +/+ = + cos = x/h = +/+ = + tan = y/x = +/+ = +	2nd quadrant: x is negative (−) y is positive (+) sin = y/h = +/+ = + cos = x/h = −/+ = − tan = y/x = +/− = −	3rd quadrant: x is negative (−) y is negative (−) sin = y/h = −/+ = − cos = x/h = −/+ = − tan = y/x = −/− = +	4th quadrant: x is positive (+) y is negative (−) sin = y/h = −/+ = − cos = x/h = +/+ = + tan = y/x = −/+ = −

Notice that sin, cos and tan are all positive in the first quadrant, but only one of them is positive in each of the other quadrants.

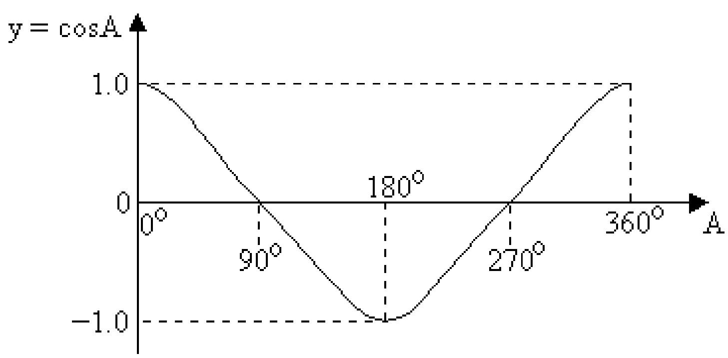
- The following ACTS as a reminder of which is positive in each quadrant:



We can work out cos and tan for angles from 0^o and 360^o , and then plot their graphs, as below:

Cosine for all angles

A	0	30	60	90	120	150	180	210	240	270	300	330	360
y=cosA	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00	-0.87	-0.50	0.00	0.50	0.87	1.00



Tangent for all angles

If you enter 90^o or 270^o in your calculator and press tan, you will get an error message. This is because as the angle approaches 90^o or 270^o the tan value heads for plus or minus infinity - so we avoid these angles in the table below:

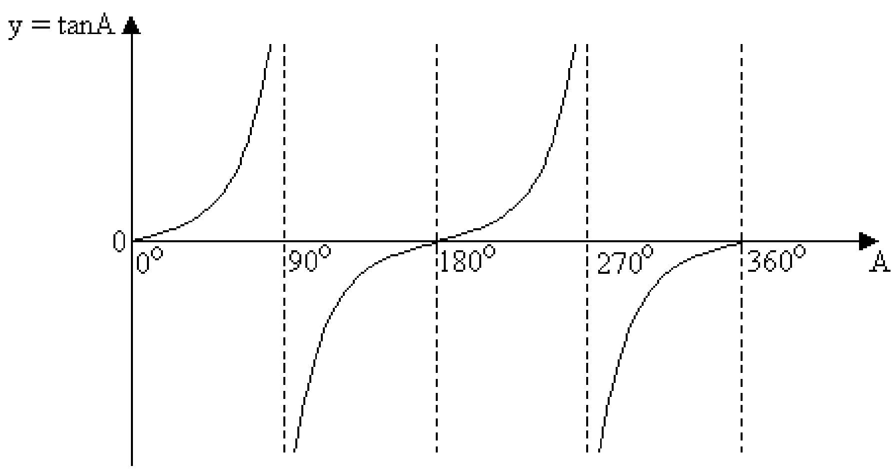
not 90° not 270°

A	0	30	60	85	95	120	150	180	210	240	265	275	300	330	360
y=tanA	0.00	0.58	1.73	11.43	-11.43	-1.73	-0.58	0.00	0.58	1.73	11.43	-11.43	-1.73	-0.58	0.00

Calculate tan for some angles close to 90° , and see how rapidly it changes - for example try:

- 89.9° , 89.99° , 89.999° , ... and
- 90.1° , 90.01° , 90.001° , ...

The graph for tan is quite different to that for sin or cos:



Solving any triangle - the sine and cosine rules

If we can work out all three sides of a triangle and all three angles, then we say that the triangle is 'solved'.

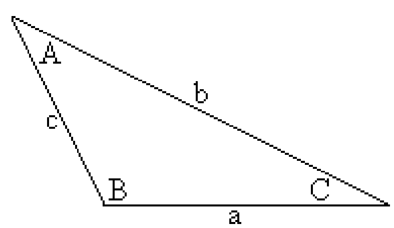
For right-angled triangles we can apply:

- Pythagoras' theorem
- the trigonometric ratios: sine, cosine or tangent

Also, if we know 2 angles in a triangle, we can find the third since, for *any* triangle:

- the sum of interior three angles = 180^o

The *sine and cosine rules* are useful because they apply to any triangle, whether it has a right-angle or not:



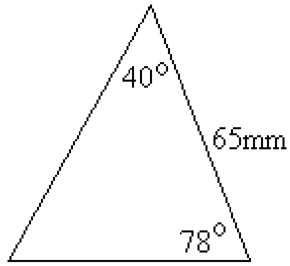
- Notice that we label each angle of the triangle and the side opposite to it with the same letter, one large and the other small

The sine rule: $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$

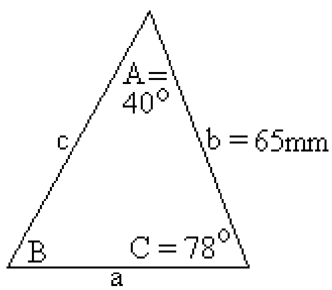
The cosine rule: $\boxed{a^2 = b^2 + c^2 - (2bc \cos A)}$

Example

Solve the following equation:



In a given problem, you can label the sides and angles any way you like, but it helps avoid confusion when using the above rules if you make sure that the side **a** and the angle **A** (etc.) are opposite to each other.



$$A + B + C = 180^\circ$$

So, $B = 180 - C - A$

$\underline{B = 62^\circ}$

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B}$

So, $a = \frac{b*\sin A}{\sin B}$

$\underline{a = 47.3\text{ mm}}$

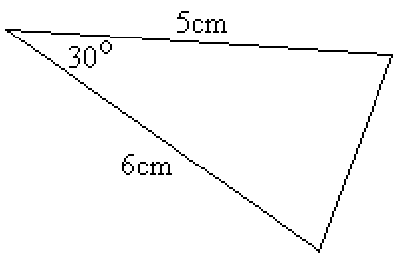
Sine rule: $\frac{c}{\sin C} = \frac{b}{\sin B}$

So, $c = \frac{b*\sin C}{\sin B}$

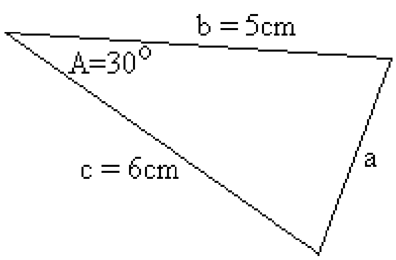
$\underline{c = 72.0\text{ mm}}$

Example

Find the unknown length in the following triangle:



If we label the unknown side as 'a', we can use the cosine rule, written as $a^2 = b^2 + c^2 - (2bc \cos A)$:



$$a^2 = b^2 + c^2 - (2*b*c*\cos A)$$
$$= 5^2 + 6^2 - (2*5*6*\cos 30)$$
$$= 9.038$$
$$a = \sqrt{9.038}$$
$$\underline{a = 3.0\text{cm}}$$

Part 3: SHAPE, SPACE & MEASURES - contents

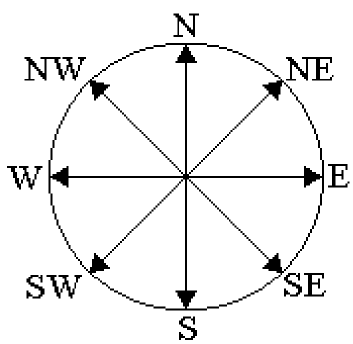
Space (Chapters 16 to 18)

Chapter 16

- BEARINGS
- ANGLES OF ELEVATION AND DEPRESSION
- SCALES
- LOCI
- TESSELLATIONS

BEARINGS - contents

The following represents points on a compass.



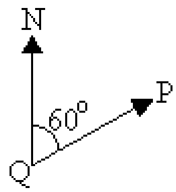
North, South, East and West (N, S, E and W) are called the cardinal directions.

A compass is useful for finding direction because its needle always point to the North Pole, and once you know that direction, you can work out the rest.

Half way between the cardinal directions are the directions NE (north east) etc.

- A *bearing* is the direction of one point relative to another, expressed as an angle

In the following, the bearing of P from Q can be expressed as N60^{o} E (said as 'north 60 degrees east'):



An commonly used alternate notation is to state a bearing as a 3 digit angle measured from the North in a clockwise direction.

- In the above case, the bearing N60^o E is the same as the bearing 060^o

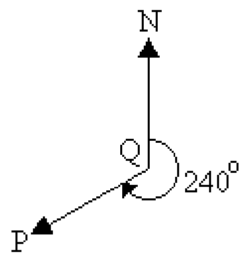
Other examples:

- N is 000^o
- E is 090^o
- SW is 225^o
- N7^o E is 007^o

If you were just given two points, such as P and Q in the last diagram, and asked for the bearing of P from Q, you could:

- draw a vertical line at Q (pointing North)
- rotate it until it passed through P
- measure or calculate the angle of rotation - this is the required bearing of P from Q

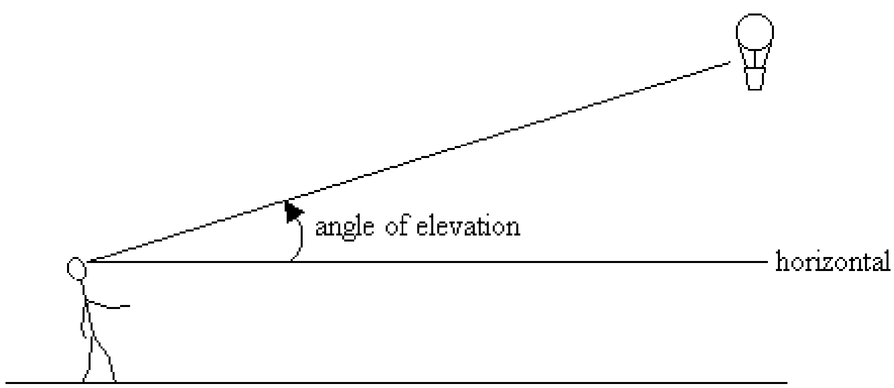
In the following, the bearing of P from Q is 240^o :



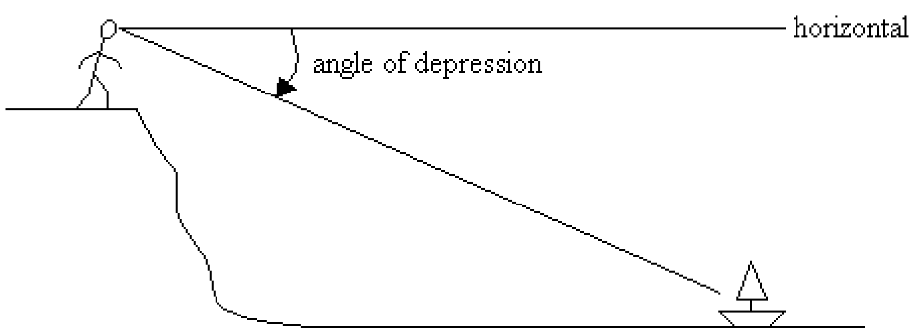
The above bearing is directly opposite to the bearing of 060^o in the previous diagram, and notice that 240^o = 60^o + 180^o .

ANGLES OF ELEVATION AND DEPRESSION - start of this chapter - contents

An angle of elevation is measured upwards from the horizontal line of sight:



An angle of depression is measured downwards from the horizontal line of sight:



SCALES - start of this chapter - contents

Models are usually made to scale, and maps are usually drawn to scale.

- The *scale* is the ratio between a length on a model or map to the corresponding length on the real thing

A half scale model of you would be a model made with a ratio of 1:2, so, for example, if you are 1.8m tall, then the model would be half of this, 0.9m tall.

Map scales are often expressed as a ratio, for example, 1:50000, or as a fraction, for example, 1/50000 (a scale of 'one to fifty thousand'). Notice that there are no units placed in the scale, so we can use any convenient unit such as cm, metres, feet or whatever. So, for example, we can interpret the scale as meaning that a distance of 1cm on the map represents a distance of 50000cm on the ground.

A scale might also be expressed as in terms of actual distance, for example, 1cm = 1km. This means that an actual one cm on the map represents one kilometre on the ground.

Example

A map has a scale of 1:50000. Two villages are 10cm apart on the map. How far apart are they in reality?

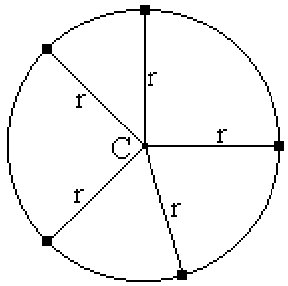
1 cm on the map represents 50000 cm on the ground.

So, 10 cm on the map represents 10*50000 cm = 500000 cm.

We divide cm by 100 to get metres. So, 500000 cm = 500000/100 m = 5000 m.

We divide metres by 1000 to get kilometres (km). So, the distance between the villages = 5000/1000 km = 5 km.

LOCI (this is plural of 'locus') - start of this chapter - contents



All the points on a circle have something in common - they are all exactly the same distance ('equidistant') from the centre. In the above diagram, all the points on the circle are a distance r, the radius of the circle, from the centre C.

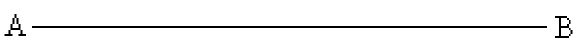
We can think of a circle as being a collection of points which satisfy the rule or condition that they are all the same distance from a fixed point.

To be general, we say that:

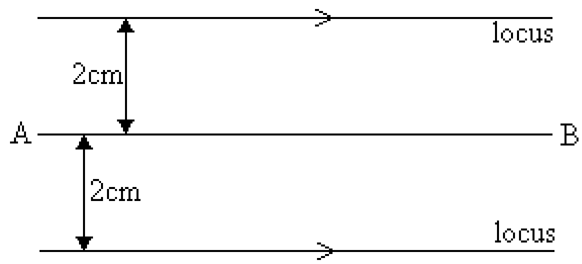
- *A locus is a set of points or lines whose location satisfies one or more conditions* - in the above case, the locus of points which are equidistant from a fixed point is a circle, with the fixed point at the centre

Example

Draw the locus which is 2cm from the straight line AB.



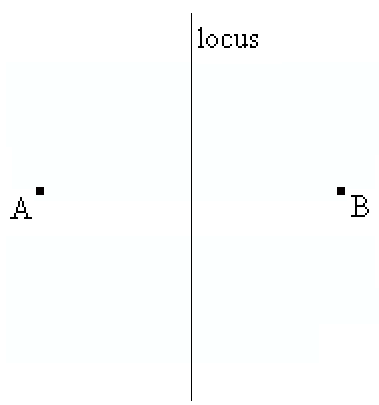
Answer:



Example

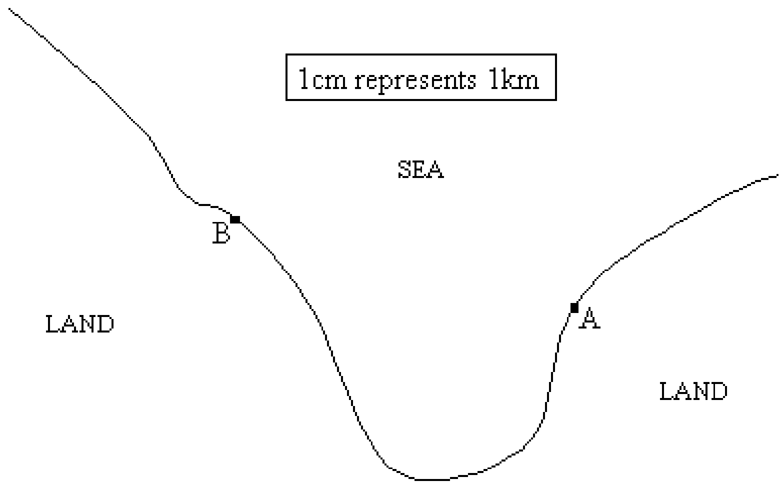
Draw the locus which is equidistant from both points A and B.

Answer:

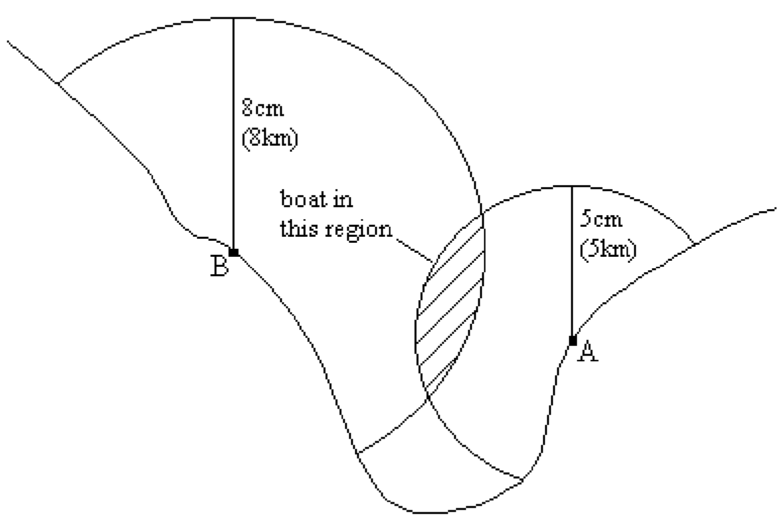


Example

A distress signal from a boat has been picked up at coastguard stations at A and B. It is estimated that the boat is within 5km of A and within 8km of B. Indicate on the diagram the region where the coastguards should concentrate their search.



- We draw an arc of radius 5cm centred on A - this is the locus of all points on the sea within 5cm (5km) of A
- We draw an arc of radius 8cm centred on B - this is the locus of all points on the sea within 8cm (8km) of B



All of the region of overlap of the circles is within 5km of A *and* within 8 km of B, so this is probably where the boat is, and so is where the coastguards should search.

2D and 3 D loci

The above are examples of two dimensional ('2D') loci, which can be draw on a sheet of paper.

However, we can also have three dimensional ('3D') loci:

- The locus of all points equidistant from a point in all directions is a sphere, with the point at the centre
- The locus of all point equidistant from a line is a cylinder, with the line at its centre

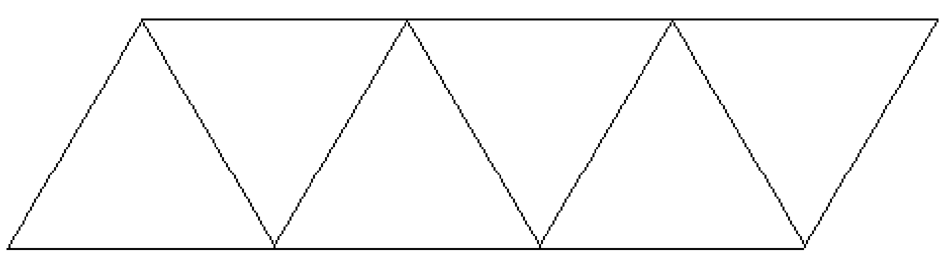
---

TESSELLATIONS - start of this chapter - contents

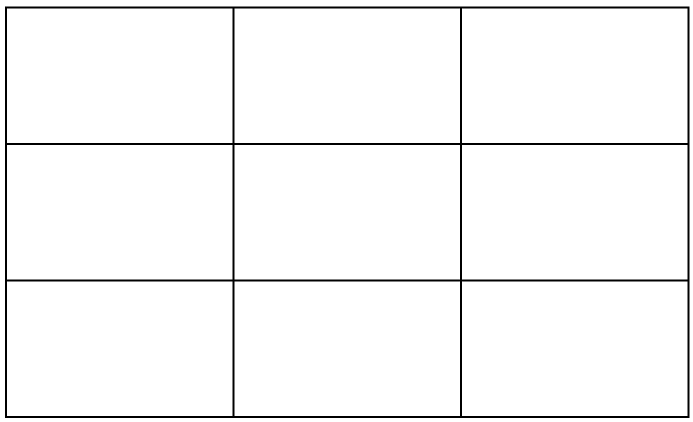
- Tessellations are patterns formed by placing identical shapes (called *congruent* shapes) such that they cover an area *without any gaps* between the shapes

For example, identical tiles can be placed to cover a surface exactly, i.e. with no gaps in between the tiles - this is practical tessellation which has been done for centuries.

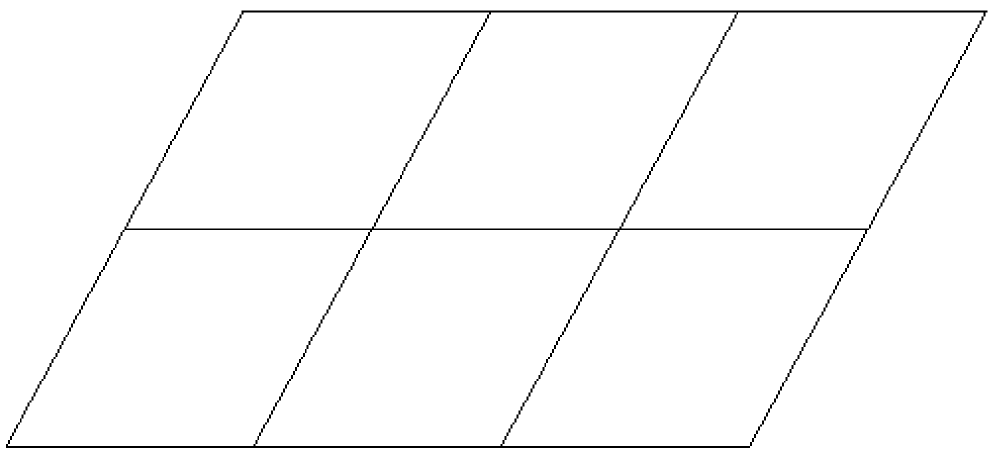
All triangles will tessellate:



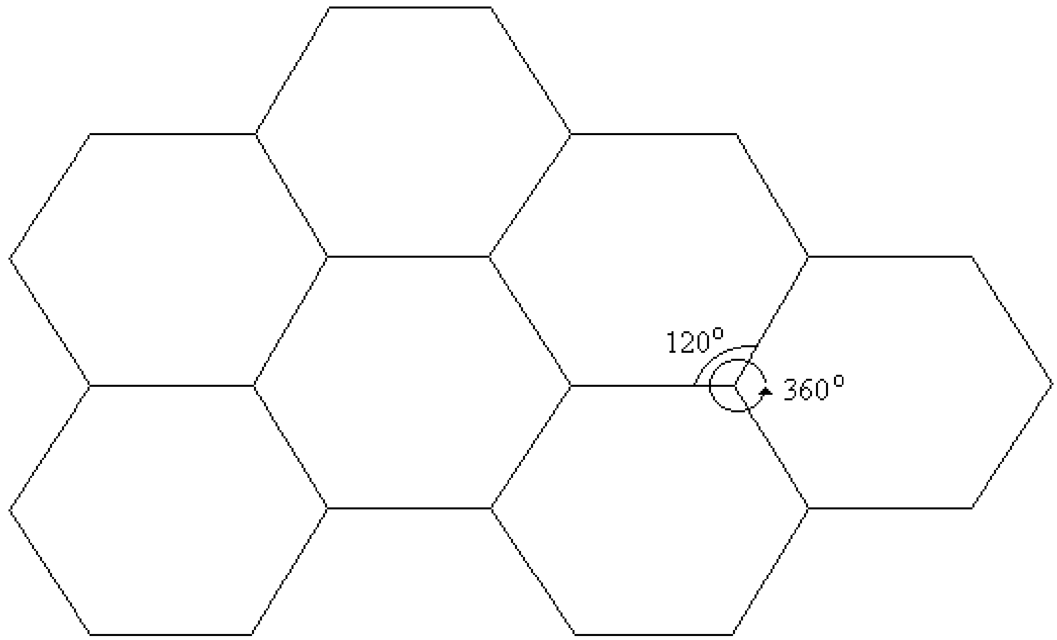
Squares and rectangles will tessellate:



Rhombuses will tessellate:



Regular hexagons will tessellate:



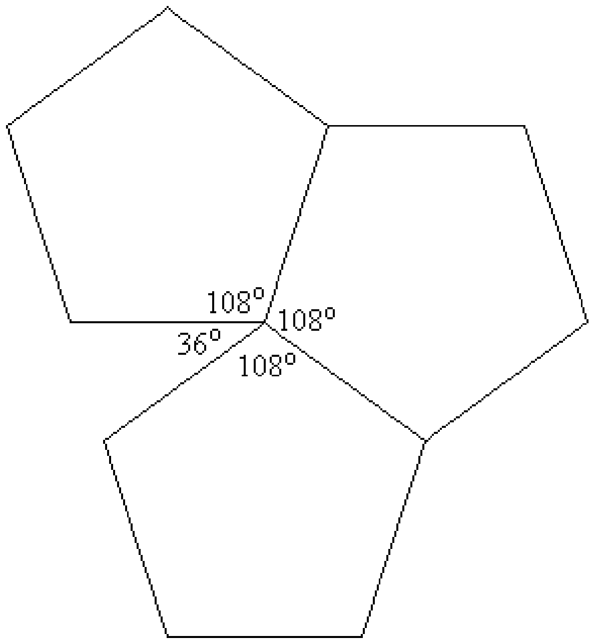
Tessellation rule for regular polygons

There are six angles inside a hexagon, called its interior angles. We see in the above diagram that the interior angles of 3 neighbouring hexagons add up to exactly 360° . If they did not, then there would either be a gap between the shapes or the shapes would overlap, i.e. they would not tessellate.

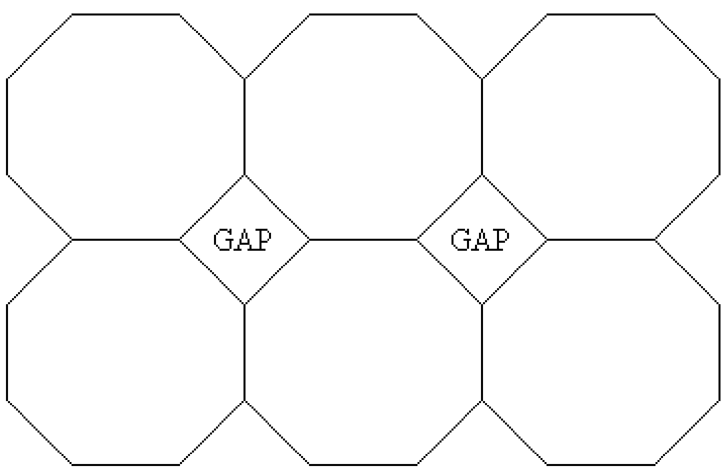
In the above case, the interior angle is 120° , and 360° /120° equals the whole number 3, which is the number of hexagons meeting at the point indicated. *This is a general rule for regular polygons* :

- Any regular polygon will tessellate if its interior angle divides exactly into 360°

Consider a regular pentagon. The interior angle of a regular pentagon is 108° , and 360° /108° = 3.3. This is *not* a whole number, so a regular pentagon will *not* tessellate:



Also, a regular octagon will not tessellate. Its interior angle is 135° and 360° /135° = 2.7.



Notice that the gaps are squares, so regular octagons *combined* with squares do tessellate (such designs are often used in tiling).

Part 3: SHAPE, SPACE & MEASURES - contents

Space (Chapters 16 to 18)

Chapter 17

- VECTORS

VECTORS - contents

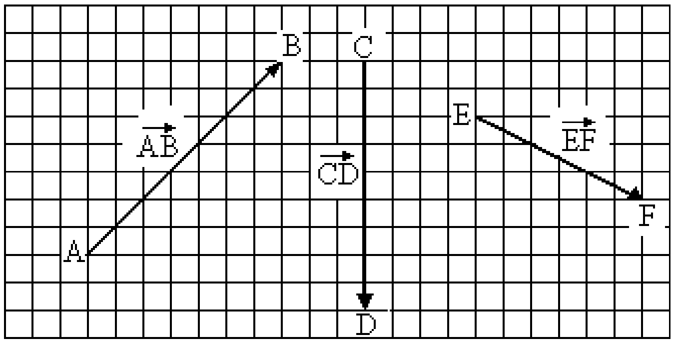
- distance is just a number of metres or yards etc., e.g. 10m or 25yards
- speed is a distance moved per second or per hour etc., e.g. 20m/s or 30mph
- displacement is a distance moved *in a particular direction* , e.g. 10 m *to the right*
- velocity is a speed *in a particular direction* , e.g. 20m/s *straight up*

The above are examples of two types of quantities - definition:

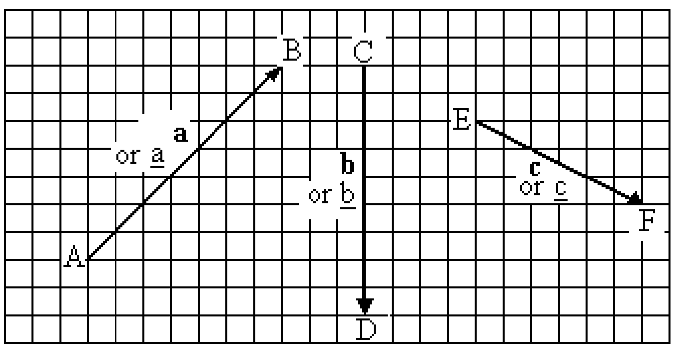
- A scalar quantity has magnitude (or size) only - for example, distance and speed
- A vector quantity has magnitude *and direction* - for example, displacement and velocity

In the following, the various vectors are displacement vectors, i.e. those which indicate movement from one place to another, but other sorts of vectors, such as velocity, have just the same properties.

In a diagram, we can represent a vector by a line with an arrow somewhere along it - for example, the following represent a displacement from A to B and from C to D and from E to F:



The letters with an arrow on top are one type of notation used to represent a vector. We also represents a vector by a letter written in **bold** or an ordinary letter <u>underlined</u> - for example:



For the first displacement vector **a** , moving directly from A to B is equivalent to moving 7 squares to the right from A along the x axis and then 7 squares up along the y axis. These values of x and y are the called the *components* of the vector **a** along the x and y axes.

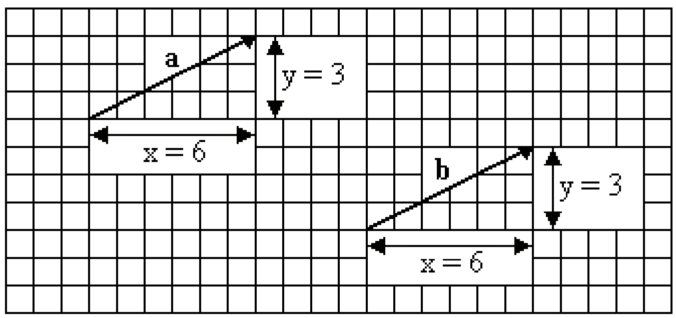
Vector **b** is composed of zero movement in the x direction, but 9 squares along the negative y axis.
Vector **c** is composed of moving 6 squares along the x axis and 3 along the negative y axis.

We can express the vectors in terms of their x and y components as:

$$\mathbf{a} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -9 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \begin{array}{l} \leftarrow \text{x component} \\ \leftarrow \text{y component} \end{array}$$

In this form we refer to the vectors as *column vectors* .

Consider the displacement vectors **a** and **b** :



Their column vectors are:

$$\mathbf{a} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

From any particular starting point, vector **a** or vector **b** would take you 6 squares in the x direction and 3 squares in the y direction, and so lead to the same final position - so the vectors are the same, so:

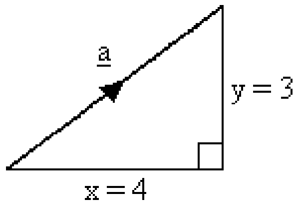
- Two vectors are the same if they have the same column vector

Magnitude (or size) of a vector

The magnitude or size of a vector is simply its length , and we can work it out from its components.

The magnitude of a vector denoted by $\overrightarrow{AB}$, $\mathbf{a}$ or $\underline{a}$ is represented by $|\overrightarrow{AB}|$, $|\mathbf{a}|$, $|\underline{a}|$ or a.

Consider the vector:



$\underline{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

The vector and its x and y components make a right-angled triangle, so we can use Pythagoras' Theorem:

The magnitude of $\underline{a}$, $|\underline{a}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

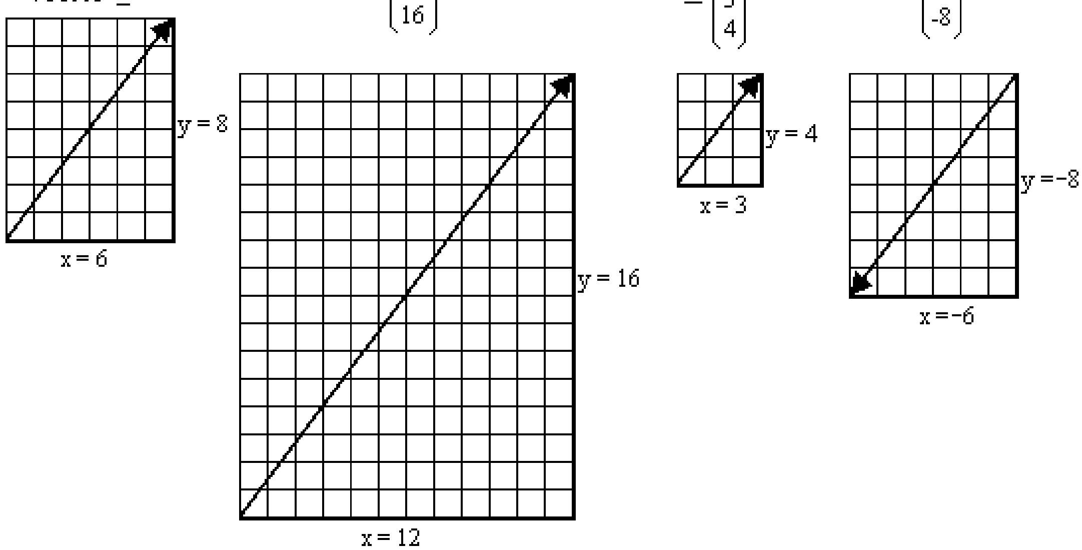
The magnitude is taken as the *positive* square root.

Multiplying a vector by a number

To multiply a vector by a number, simply multiply each component of its column vector by the number:

$\underline{a} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ $\quad 2*\underline{a} = \begin{pmatrix} 2*6 \\ 2*8 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$ $\quad \frac{1}{2}*\underline{a} = \begin{pmatrix} 0.5*6 \\ 0.5*8 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad -\underline{a} = \begin{pmatrix} -1*6 \\ -1*8 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$

vector $\underline{a}$



Notice that the vectors are all parallel to each other, and that -$\underline{a}$ is in the opposite direction to $\underline{a}$.

Working out the magnitudes we get:

$$|\underline{a}| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

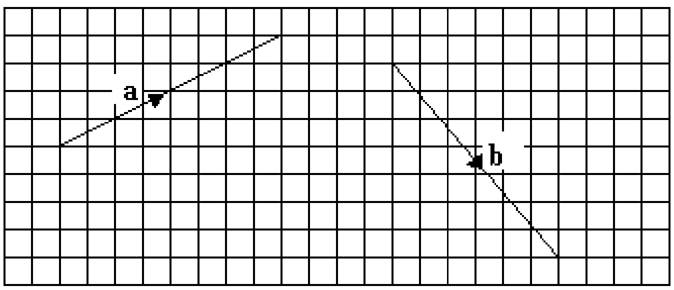
$$|2*\underline{a}| = \sqrt{12^2 + 16^2} = \sqrt{400} = 20 \quad \longleftarrow \text{ 2 * magnitude of } \underline{a}$$

$$|\tfrac{1}{2}*\underline{a}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \quad \longleftarrow \text{ 1/2 magnitude of } \underline{a}$$

$$|-\underline{a}| = \sqrt{(-6)^2 + (-8)^2} = \sqrt{100} = 10 \quad \longleftarrow \text{ same as magnitude of } \underline{a}$$

Addition

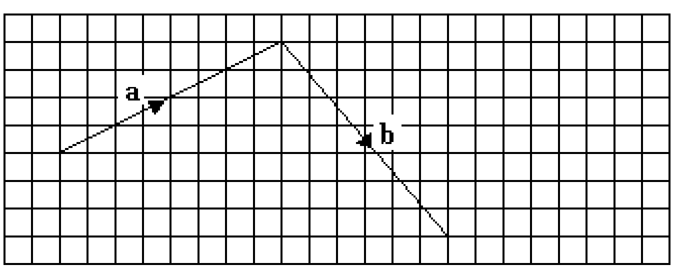
Suppose that we have the following two vectors, and want to add them:



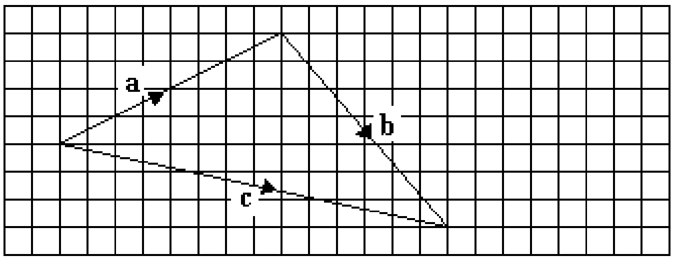
As column vectors:

$$\mathbf{a} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$

In a real situation, if these were displacement vectors, adding them would mean that, from a certain position, you would move the distance and direction indicated by **a** and *then* the move the distance and direction indicated by **b** . So adding them means following each one in succession - which means that we just join the vectors nose-to-tail:



If we join the start of **a** to the end of **b** , we get a new vector, **c** . We can see that following the vector **c** is the same as following **a** and then **b** , so **c** is the result of adding **a** and **b** , and is called their *resultant* vector. To produce the column vector of **c** we add the x and y values of the column vectors of **a** and **b** :



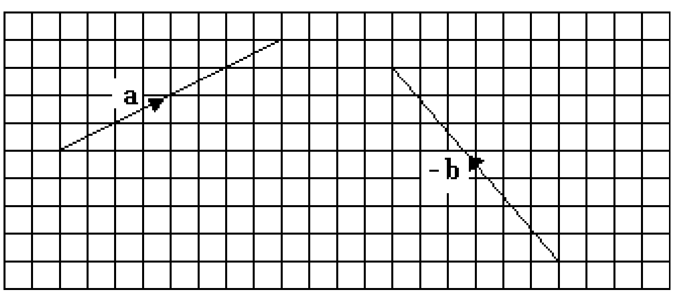
$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$
$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$
$$= \begin{pmatrix} 8+6 \\ 4-7 \end{pmatrix}$$
$$\mathbf{c} = \begin{pmatrix} 14 \\ -3 \end{pmatrix}$$

Subtraction

Suppose that we start with the same two vectors as in the above addition but this time:

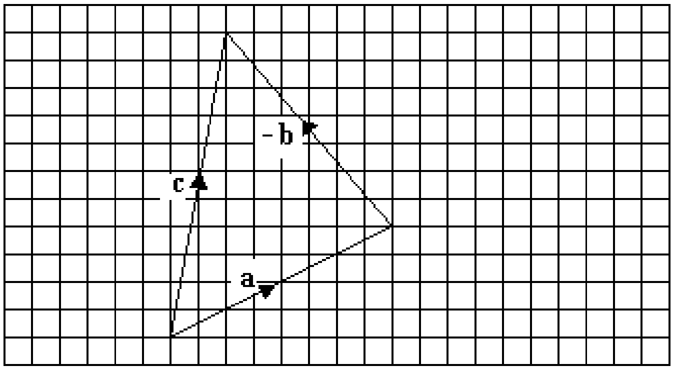
- we want to determined **a - b**

Now **a - b** is the same as **a** + (**-b**). We know from earlier that **-b** is a vector the same as **b** but opposite in direction:



$$\mathbf{a} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$
$$-\mathbf{b} = -1 * \begin{pmatrix} 6 \\ -7 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$$

We now *add* these as before:



$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$
$$= \mathbf{a} + (-\mathbf{b})$$
$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} 8-6 \\ 4+7 \end{pmatrix}$$
$$\mathbf{c} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$$

Part 3: SHAPE, SPACE & MEASURES - contents

Space (Chapters 16 to 18)

Chapter 18

- TRANSFORMATIONS

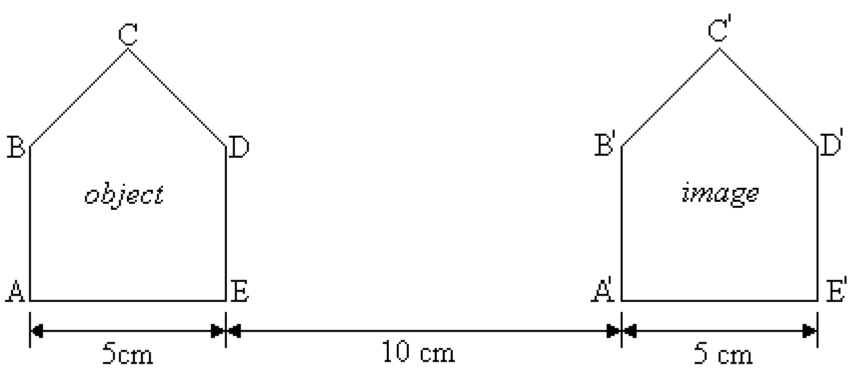
TRANSFORMATIONS - contents

To transform (or 'map') something involves some sort of change. The following are all types of transformations:

1. Translations

This simply means that the original figure is moved from one place to another (slid left, right, up or down).

Consider:



The original figure, the 'object', has been moved to the right, producing the 'image'. To find the distance moved, look at one point on the object and the corresponding point on the image. For example, point A has moved 15cm to become point A'. All the points have moved the same distance.

To describe a translation, we state:

1. the distance moved, and
2. the direction moved

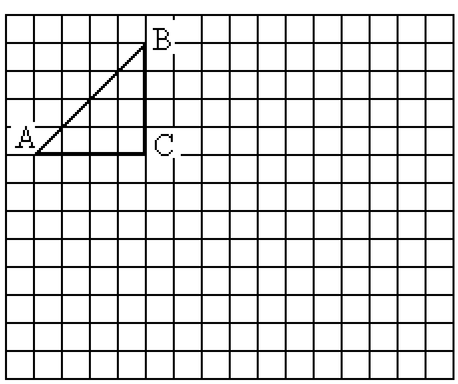
Thus, the above translation is 15cm to the right. To describe the transformation fully we would say:

- the transformation which maps figure ABCDE to figure A'B'C'D'E' is a translation of 15cm to the right

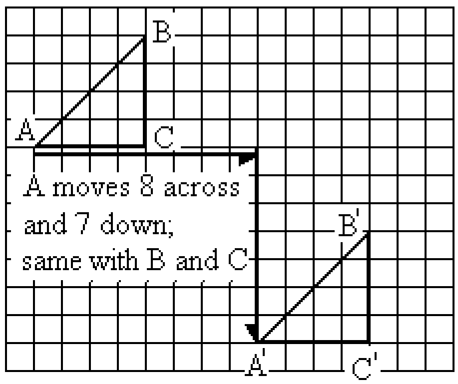
Since a translation is a distance moved in a certain direction, *we can represent a translation by a vector* .

Example

Perform the following translation of the triangle ABC: $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$



The vector means that each point on the object is moved 8 squares in the x direction (right) and 7 squares in the negative y direction (down). This is done to each of A, B and C to locate A', B' and C', which are then joined together:



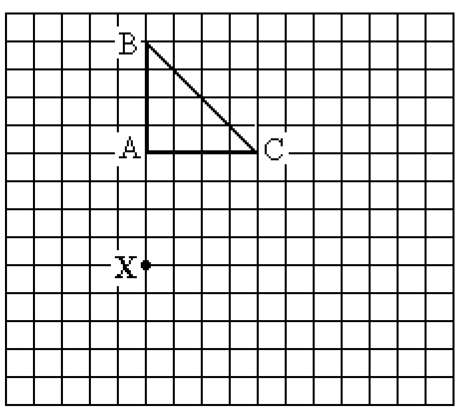
2. Rotations

To describe a rotation, we state:

1. the centre of rotation
2. the angle of rotation
3. the direction of rotation (clockwise or anticlockwise)

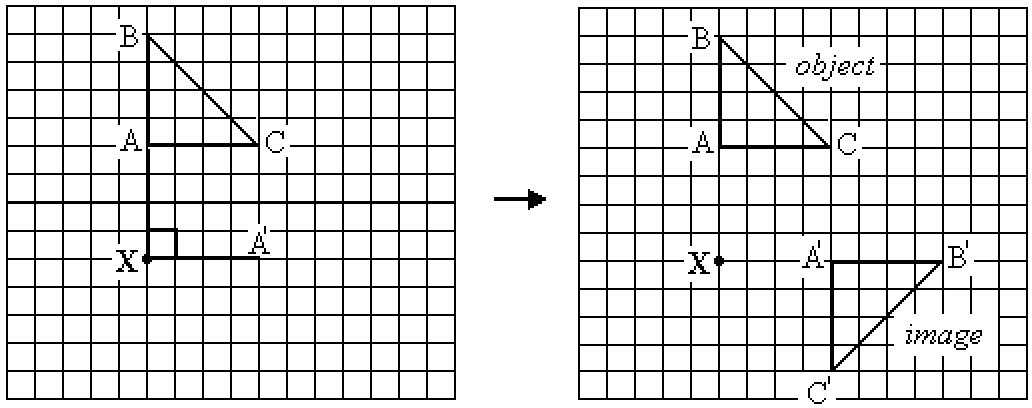
Example

Rotate the following triangle by 90^o clockwise, about the centre X:



From the original vertices A, B and C (a vertex is where two lines meet) we need to locate the new vertices A', B' and C'.

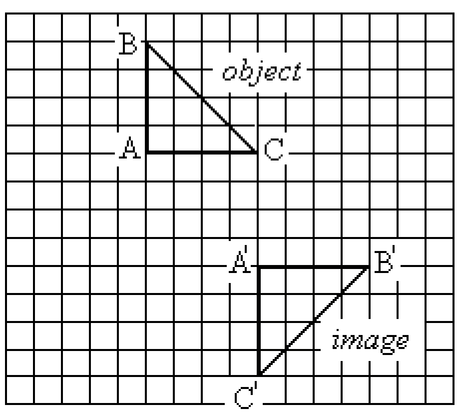
From X, we draw a line XA. We rotate this line by 90^o clockwise, and mark A', such that XA' = XA.
We repeat this to locate B' and C', and then complete the new triangle:



You may be given two figures, the original object and its image, and asked to describe the rotational transformation that maps the object to the image. The method can be illustrated using the result of the previous example:

Example

Describe fully the transformation that maps ABC to A'B'C':



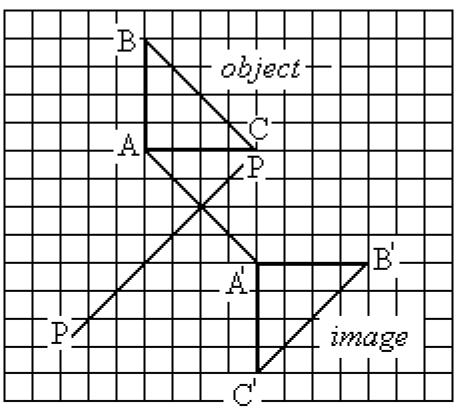
We need to state the direction and angle of rotation, and the centre of rotation.

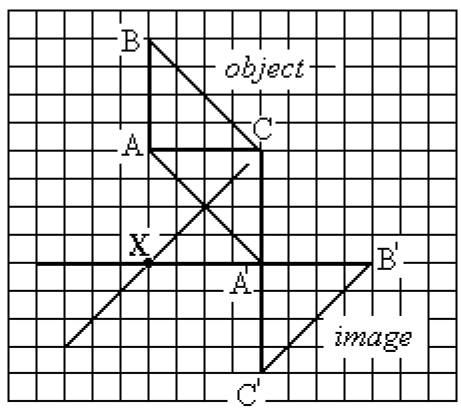
The direction and angle are found by comparing two corresponding edges. We see that AB has been rotated by 90^o clockwise to produce A'B'.

To find the centre of rotation:

- join two corresponding points with a straight line and draw the perpendicular bisector to the line (i.e. a line in the middle, at 90^o) - in the 1st dia. below, the line AA' has been drawn, and the perpendicular bisector PP drawn to this
- repeat this with two other corresponding points - this has been done with CC' in the 2nd diagram below

The centre of rotation X is the point where the two perpendicular bisectors cross:





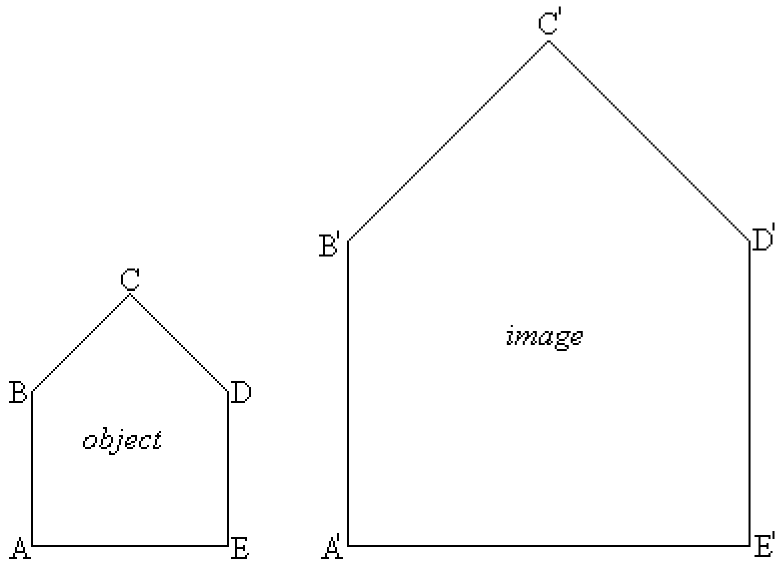
To describe the transformation fully we would say:

- the transformation which maps figure ABC to figure A'B'C' is a rotation of 90^o clockwise about the centre of rotation X

3. Enlargements

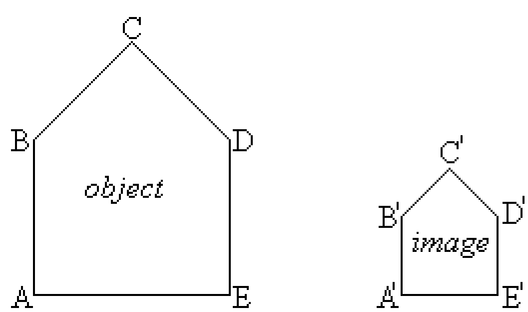
When we enlarge something we change its size.

In the following the figure ABCDE has been enlarged by a *scale factor* of 2:



Every edge has been doubled in length, but all the angles are still the same. Thus, the image is *similar* to the object.

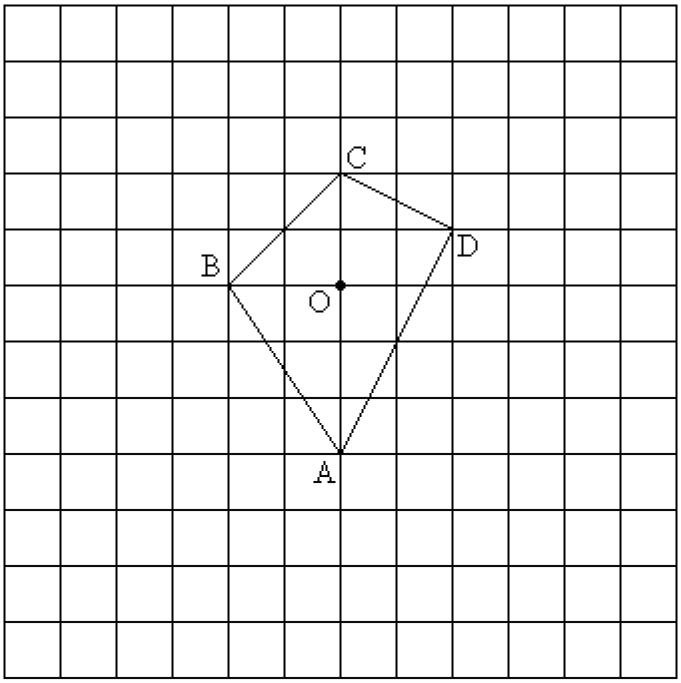
Though the word 'enlarge' means 'to make bigger', we also use the same term when the scale factor is less than 1, in which case the image is smaller than the object - for example, applying a scale factor of 1/2 to the above object gives:



Another way of making an enlargement is by using a *scale factor* and a *centre of enlargement* :

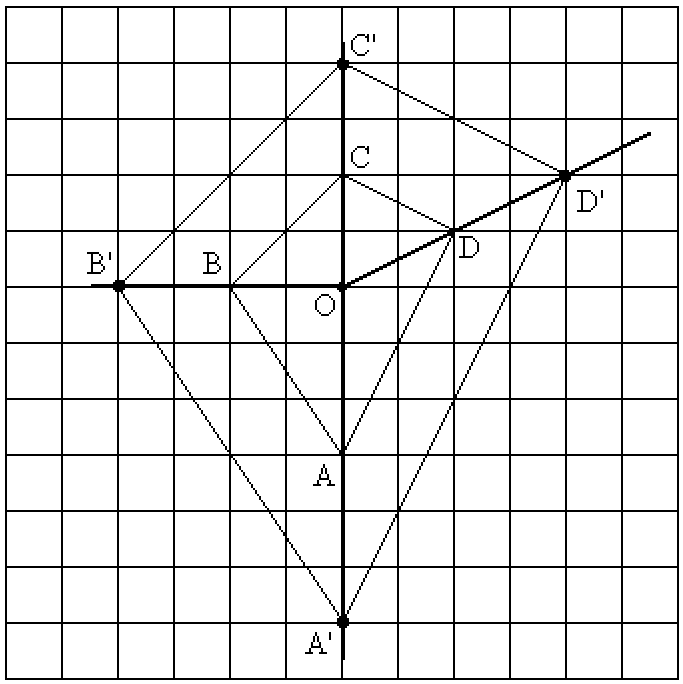
Example

Produce the enlargement of the figure below using the centre O, and scale factor 2:



Procedure:

1. From O draw straight lines through each vertex
2. Mark A' at a distance 2*OA from O
3. Repeat (2) for all the vertices
4. Join the points A' to B' etc. to produce the image



Object (cm)		Image (cm)	
OA	3	OA'	6
OB	2	OB'	4
OC	2	OC'	4
OD	2.2	OD'	4.4

It may be helpful to produce a table as above. But if the object is on a grid, it is possible to move the appropriate number of positions in the x and y directions to find each new point. For example, D is 2 right and 1 up from O, so D' is 4 right and 2 up from O.

Note: The centre of enlargement can be anywhere, inside or outside the object, but the procedure is the same.

Also,

$$\boxed{\text{scale factor} = \frac{\text{distance of a point on the image from O}}{\text{distance of the corresponding point on the obect from O}}}$$

Or,

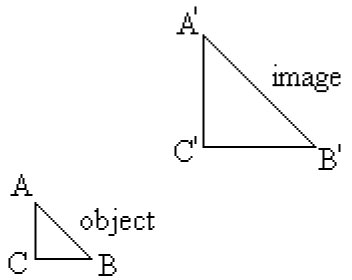
$$\boxed{\text{scale factor} = \frac{\text{image length}}{\text{corresponding object length}}}$$

must use same units on top and bottom

The next example illustrates the reverse procedure - finding the scale factor and centre of enlargement.

Example

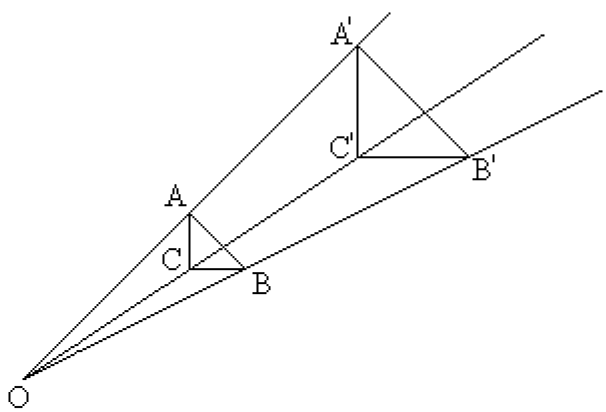
Find the scale factor and the centre of enlargement of the following, given that AB = 1cm and A'B' = 2cm:



$$\text{scale factor} = \frac{\text{image length}}{\text{corresponding object length}}$$

So, $\text{scale factor} = \frac{A'B'}{AB} = \frac{2\text{cm}}{1\text{cm}} = 2$

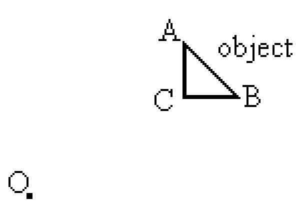
Now, we draw lines though corresponding vertices - they cross at the centre of enlargement, O:



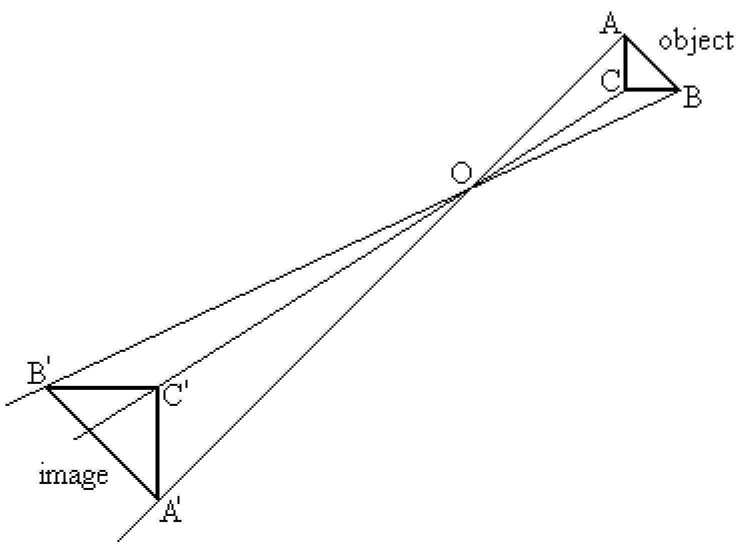
The next example indicates the meaning of a negative scale factor:

Example

Draw the image of the following object using the centre of enlargement O, with a scale factor of -2:

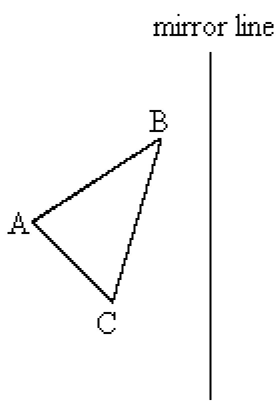


- When a scale factor is negative, the image is on the opposite side of the centre of enlargement



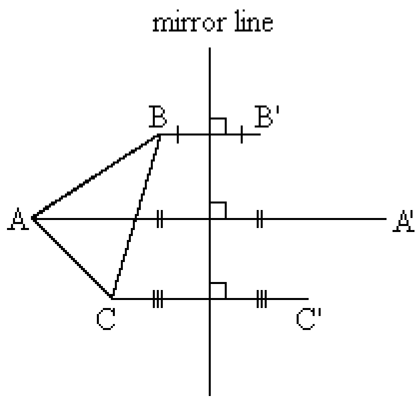
4. Reflections

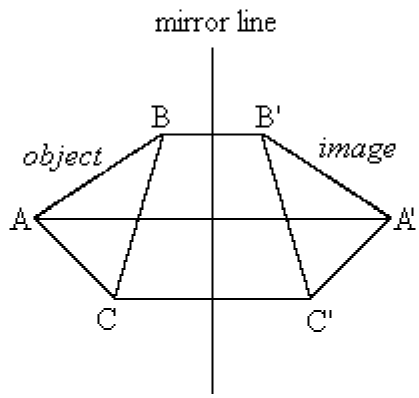
Here we reflect the triangle ABC about the mirror line indicated:



The mirror line represents a mirror seen edge-on, so it just appears as a straight line. The image is formed just as it would be in a real mirror. The procedure is:

1. From point A, draw a perpendicular to the mirror, and extend it beyond the mirror to produce the image point A'. The image point A' is as far behind the mirror as the object point A is in front
2. This is repeated to produce an image point from each object point - as in the left-hand diagram below
3. The image points are now joined to produce the image - as in the right-hand diagram below

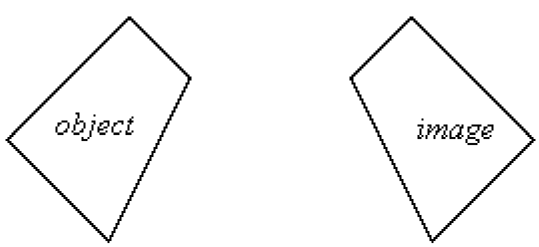




The object and image are said to be *oppositely congruent* - if you cut them out, and *turn one over* , it exactly fits on the other.

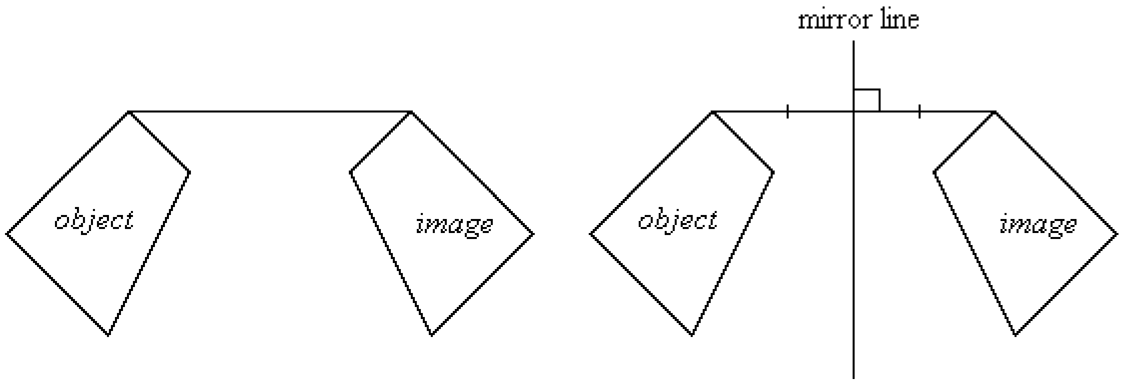
Example

Locate the mirror line for the object and image below.



The procedure is:

1. Draw a line between two corresponding points on the object and the image (left-hand diagram below)
2. Draw a line in the *middle* of the line and at 90^o to it - this is called the perpendicular bisector, and is the required mirror line



Part 3: SHAPE, SPACE & MEASURES - contents

Measures (Chapters 19 and 20)

Chapter 19

- UNITS
- AREAS OF PLANE SURFACES
- VOLUMES OF CUBES AND CUBOIDS

UNITS - contents

It would not mean very much to say that a particular piece of string is '5 long'. We need to say whether we mean 5 inches, 5 yards, 5 cm, 5 metres, 5 miles or whatever. Inches, yards, cm, etc. are all *units* of length.

For many year Britain has used *Imperial units* . These include:

- inches and feet for length
- pints and gallons for volume
- ounces and pounds for mass (or weight)

Europe uses the metric system, and this is now used more and more in Britain.

In the metric system the units are based on multiples of 10.

Mass

$$\begin{aligned}\text{one tonne} &= 1000 \text{ kilograms (kg)}\\ \text{one kilogram} &= 1000 \text{ grams (g)}\\ \text{one gram} &= 1000 \text{ milligrams (mg)}\end{aligned}$$

A small teaspoon of water has a mass of about 1 gram.

One tonne is also called a 'metric tonne' (it is not the same as the Imperial ton used in Britain)

Length

One centimetre is about the width of a little finger. One kilometre is about half a mile.

$$\begin{aligned}\text{one kilometre (km)} &= 1000 \text{ metres (m)}\\ \text{one metre} &= 100 \text{ centimetres (cm)}\\ \text{one centimetre} &= 10 \text{ millimetres (mm)}\end{aligned}$$

Note: 1 m = 100 cm = 100*10 mm = 1000 mm.

Volume (or capacity)

$$\text{one litre}\,(l) = 1000 \text{ millilitres } (\text{m}l)$$
$$\text{one centilitre}\,(\text{c}l) = 10 \text{ millilitres}$$

Note: An italic *l* has been used above, simply so that it does not look like a number 1 (many typewriters and computers use the number '1' to stand for the small letter '*l* ').

A mass of 1 g of water has a volume of 1 m*l* , so a small teaspoon of water is about 1 m*l* of water.

Compound measures arise when a quantity involves more than one unit. For example, by definition:

$$\boxed{\text{density} = \frac{\text{mass}}{\text{volume}}}$$

So, if a mass of 1 g of water has a volume of 1 m*l* , then the density of water is 1g/m*l* (one gram per millilitre).

Approximate equivalents between Imperial and metric units

- 1kg is about 2.2 pounds
- 8km is about 5 miles
- 1 metre is about 39 inches
- 1 foot is about 30.5cm
- 1 litre is about 0.2 gallons

Example

The Chancellor of the Exchequer has just increased petrol by 4 pence a litre. Approximately how much is that per gallon?

- 1 litre is about 0.2 gallons, so
- 5 litres is about 1 gallon.

So, we multiply the price increase by 5. The increase per gallon is approximately 5 * 4 = 20 pence

Example

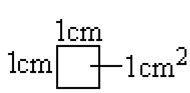
The speed limit of a certain road in France is 80 km per hour. What is that approximately in miles per hour?

- 8km is about 5 miles, so
- 80 km is about 50 miles, so
- 80 km per hour is about 50 miles per hour

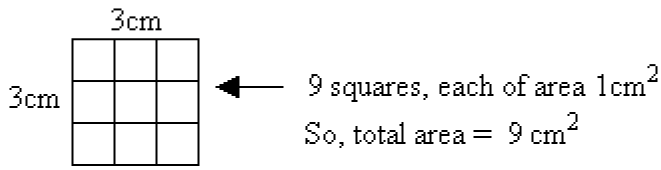
AREAS OF PLANE SURFACES ('plane' means 'flat') - start of this chapter - contents

Squares and rectangles

A 1cm by 1cm square has an area of $1cm^2$ ('one square cm').

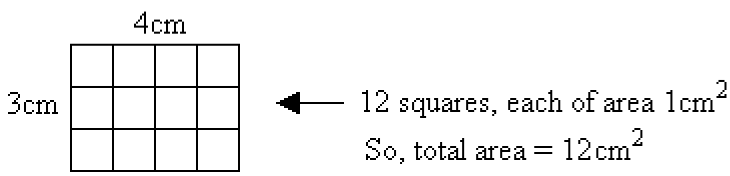


A square 3cm by 3cm is made up of nine $1cm^2$:



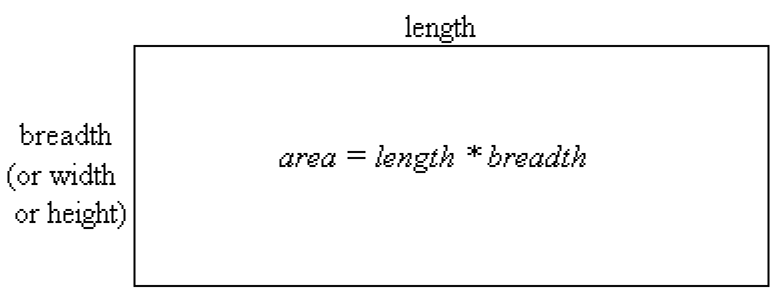
We see that the area equals 3cm * 3cm = 9cm^2 .

A rectangle 3cm by 4cm is made up of twelve 1cm^2 :



We see that the area equals 3cm * 4cm = 12cm^2 .

In general, the area of a square or rectangle is found using:

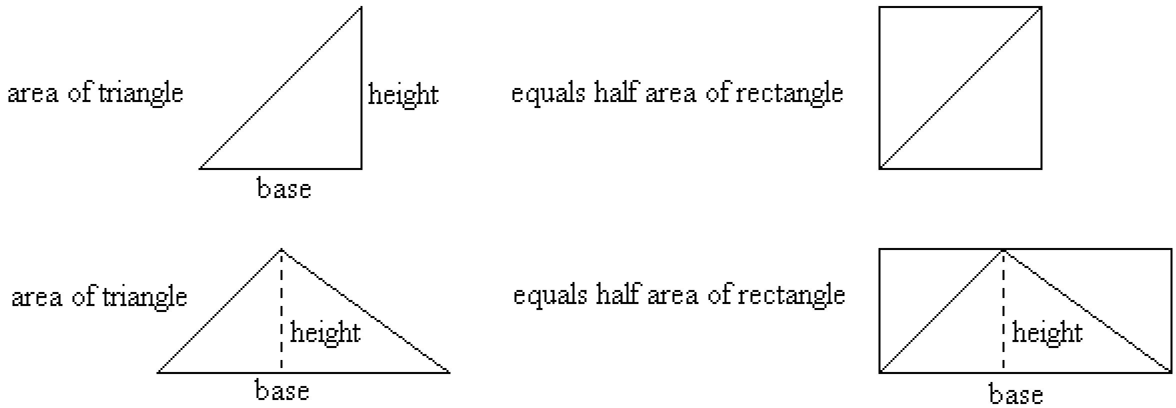


Note:

- the units must be the same for both length and breadth - e.g. both in cm gives area in cm^2
- the order doesn't matter: length * breadth = breadth * length

Triangles

The area of any triangle equals half the area of a suitable rectangle:

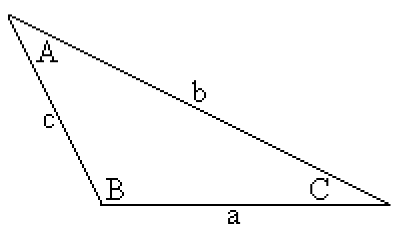


Thus, we can express the area of a triangle in terms of the area of a rectangle:

area of rectangle = base * height

So, $\boxed{\text{area of triangle} = \frac{1}{2} * \text{base} * \text{height}}$

However, sometimes the vertical height may not be known - in which case there is an alternate equation that can be used. We can find the area of any triangle if we know the length of two sides and the included angle, i.e. the one between them:



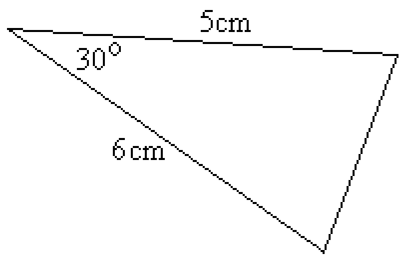
$$\boxed{\text{area of a triangle} = \frac{1}{2} * (\text{product of two sides}) * (\text{sine of the included angle})}$$

For the above triangle, we can express this as:

$$\begin{aligned} \text{area} &= \frac{1}{2}\, ab \sin C \\ &= \frac{1}{2}\, ac \sin B \\ &= \frac{1}{2}\, bc \sin A \end{aligned}$$

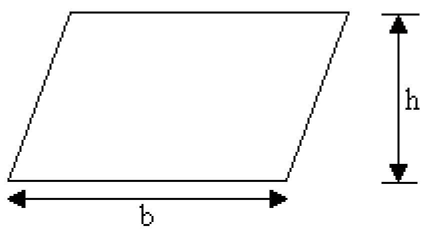
Example

Find the area of the triangle:

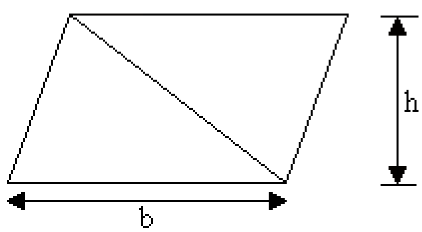


area = 0.5 * 5 * 6 * sin30 = 7.5 cm^2

Parallelograms



If we draw a diagonal (in either direction) across the parallelogram, we get two triangles, and we already have a formula for the area of a triangle:

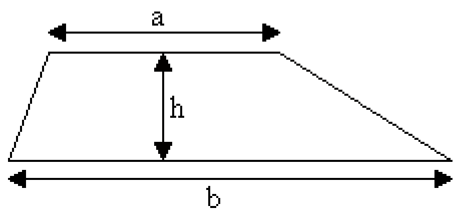


For both triangles, area $= \frac{1}{2}$ base * height $= \frac{1}{2}$ b h

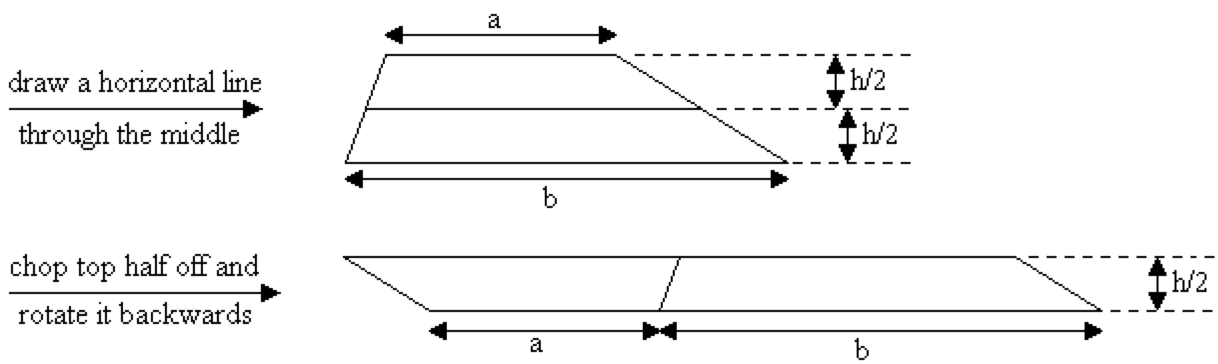
So, total area of the parallelgram $= 2 * \frac{1}{2}$ b h = b h

So, $\boxed{\text{area of a parallogram = base * height = b h}}$

Trapeziums



From above, we already know that the area of a parallelogram equals base*height, and so if we can change the trapezium into a parallelogram, we can find its area:



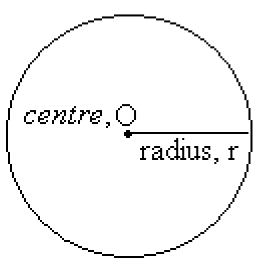
The overall shape is now a parallelogam, so its area = bass * height

$= (a+b) * (h/2)$

So, $\boxed{\text{area of trapezium} = \frac{1}{2}(a+b)\,h}$

Circles

Figures do not have to have straight edges in order to have areas.

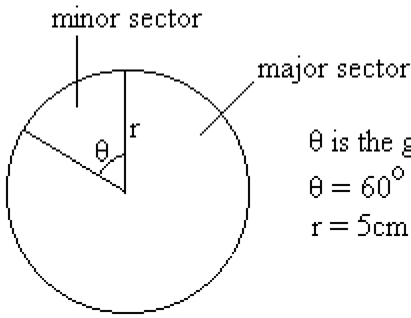


$\boxed{\text{area of a circle} = \pi\, r^2}$

π (said as 'pie') = 22/7 = 3.14 (it may be on your calculator)

Example

Find the total area of the following circle, and the areas of the major and minor sectors:



θ is the greek letter 'theta', and is often used to represent an angle

$\theta = 60^{\circ}$

$r = 5\text{cm}$

$$\begin{aligned}\text{area of circle} &= \pi r^2 \\ &= \pi * 5^2 \\ &= \underline{78.5\ \text{cm}^2}\end{aligned}$$

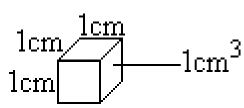
The total angle in a circle is 360°, so

$$\text{area of minor sector} = \frac{60}{360} * 78.5 = \underline{13.1\ \text{cm}^2}$$

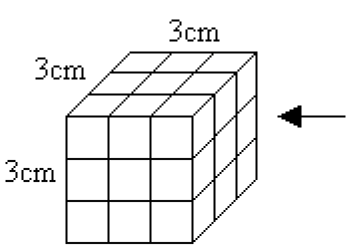
$$\text{area of major sector} = 78.5 - 13.1 = \underline{65.4\ \text{cm}^2}$$

VOLUMES OF CUBES AND CUBOIDS - start of this chapter - contents

A 1cm by 1cm by 1cm cube has a volume of 1cm^3 ('one cubic cm').



A *cube* 3cm by 3cm by 3cm is made up of twenty seven 1cm^3 :

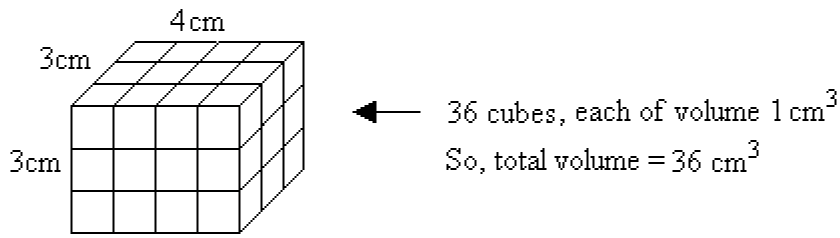


27 cubes, each of volume 1 cm^3

So, total volume = 27 cm^3

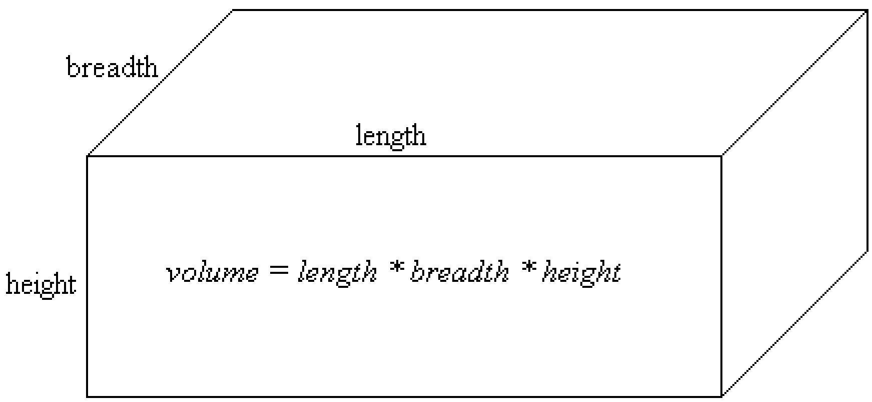
We see that the volume equals 3cm * 3cm * 3cm = 27cm^3 .

A *cuboid* 3cm by 3cm by 4cm is made up of thirty six 1cm^3 :



We see that the volume equals 3cm * 3cm * 4cm = 36 cm^3 .

In general, the volume of a cube or cuboid is found using:

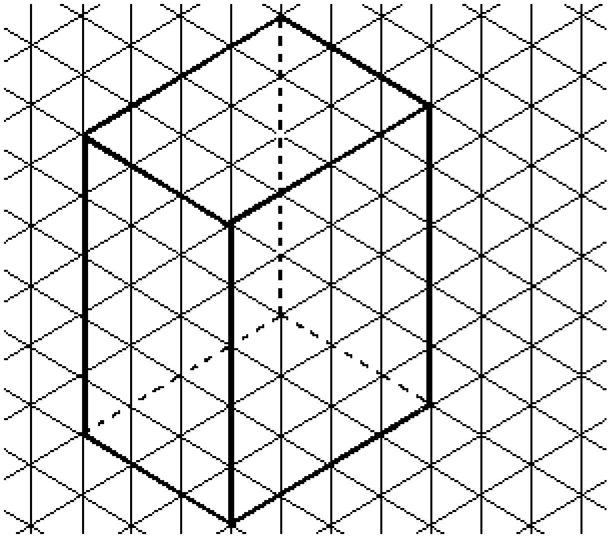


Note:

- the units must be the same for all of length, breadth and height - e.g. all in cm gives volume in cm^3
- the order that length, breadth and height are multiplied does not matter

Use of isometric paper

Isometric paper is used by, for example, engineers, to help represent 3D (3 dimensional) objects in 2D. The paper is made up tessellations of equilateral triangles. The following represents a cuboid drawn on isometric paper:



GCSE Maths Notes

Part 3: SHAPE, SPACE & MEASURES - contents

Measures (Chapters 19 and 20)

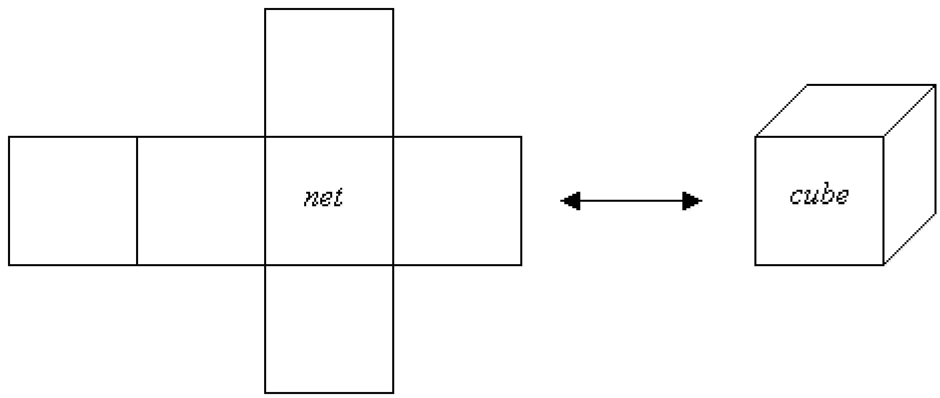
Chapter 20

- AREAS AND VOLUMES OF VARIOUS 3D OBJECTS
- DIMENSIONS AND FORMULAE

AREAS AND VOLUMES OF VARIOUS 3D OBJECTS - contents

Cubes

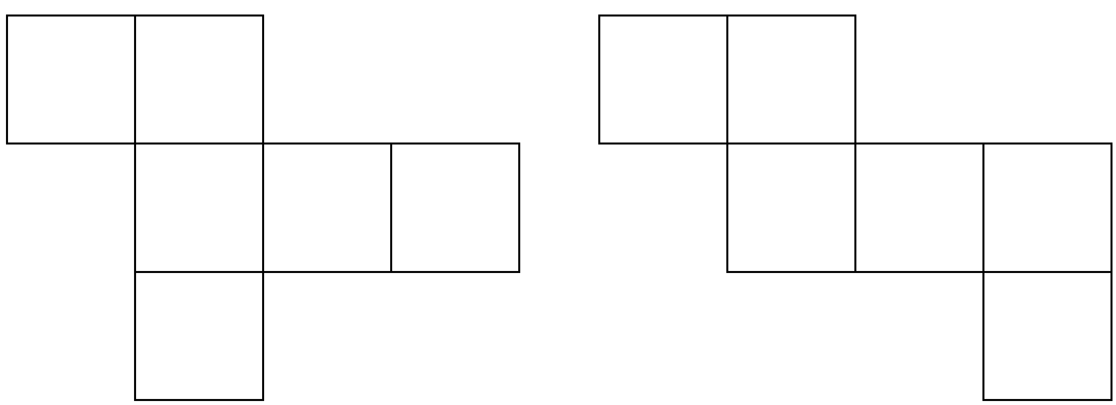
The cross shape below is called a **net** of the cube. The net can be cut out and folded to form the cube:



The surface area of the cube is equal to the surface are of its net, so:

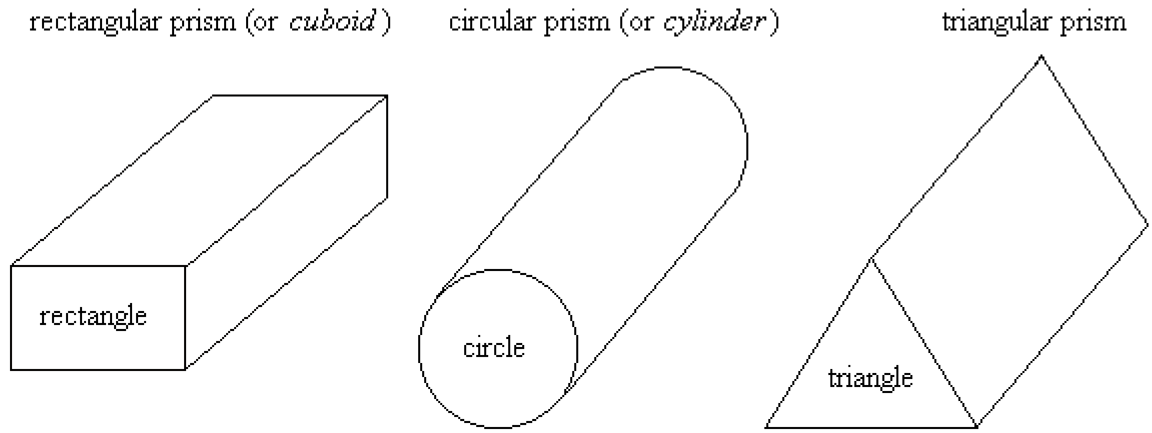
- surface area of a cube = 6 * area of one face of the cube

A cube can also have other nets, but they also have the same surface area as the cube:

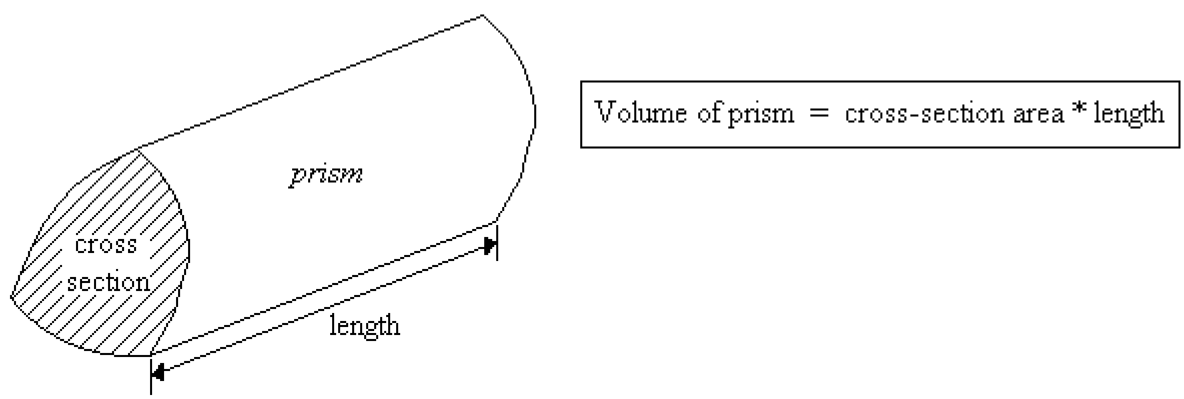


Prisms

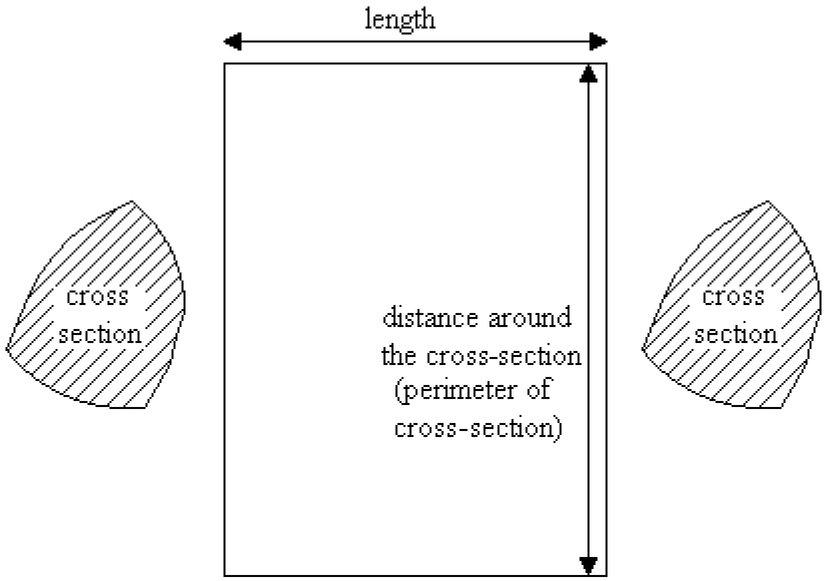
A prism is any solid with a uniform (=constant, unchanging) cross-section.
The name of a prism is usually taken from the shape of its cross-section:



We can work out the volume and surface area of a prism with *any* shape cross-section the same way, so we use the following diagram to represent *any* prism:



The net of the above prism is:



The surface area equals the two cross sections added, plus the area of the rectangular part. So, in general:

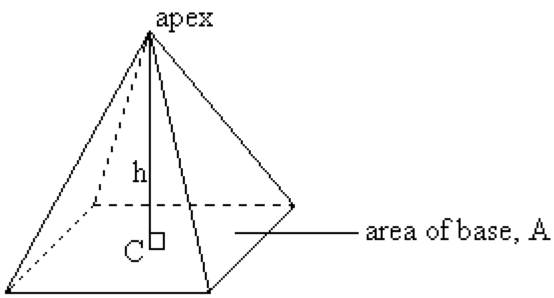
surface area of a prism = (2* area of cross-section) + (perimeter of cross-section* length)

Pyramids

A pyramid has a polygon for a base, and sloping triangular sides which meet at a point (the apex).

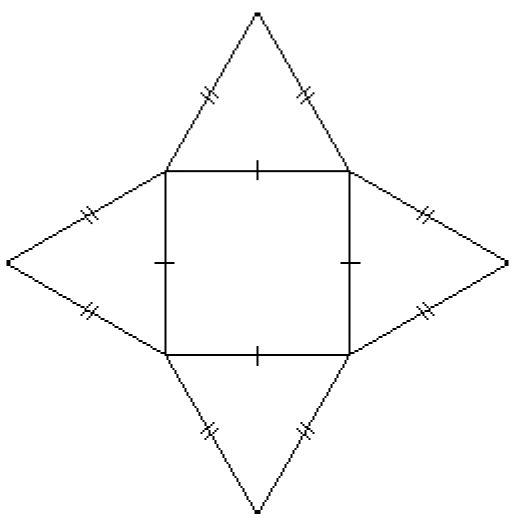
In a 'right pyramid', the apex is directly over the centre C of the base (in a 'skew pyramid', it is not).

The right pyramid is the one we usually mean when we simply refer to a 'pyramid'.



$$\text{volume of a pyramid} = \frac{1}{3} * \text{area of base} * \text{perpendicular height} = \frac{1}{3} A h$$

If the pyramid were made of cardboard, we could just cut along its edges and lay it flat to produce its net. The net of a square-based pyramid looks like:



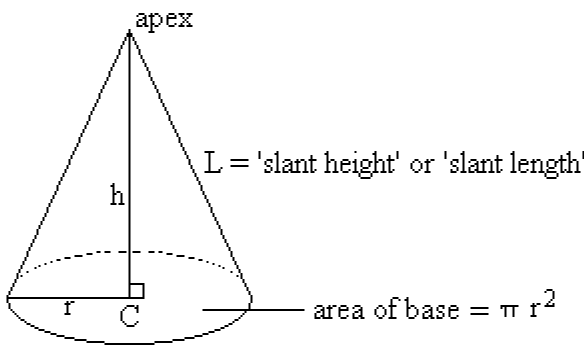
$$\text{surface area of a pyramid} = \text{area of base} + \text{areas of triangular faces}$$

Notice that for the square-based pyramid the triangles are congruent (identical).

Cones

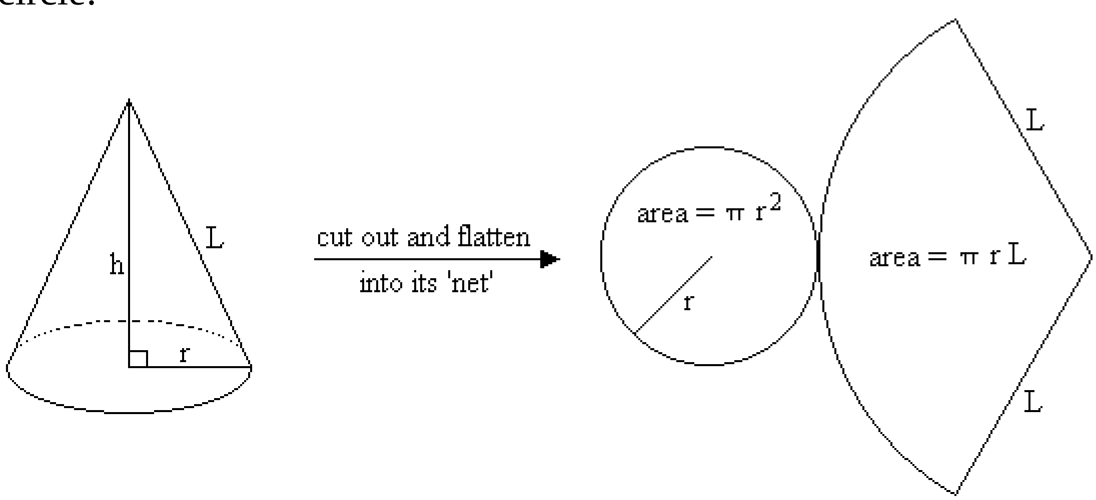
A 'circular cone' has a circular base.

As with pyramids, the 'right cone' has its apex directly over the centre of the circle.



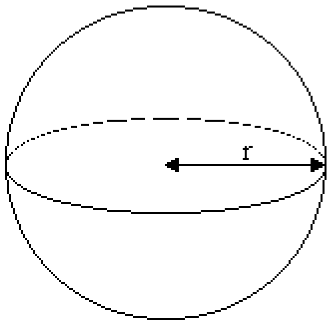
$$\text{volume of a cone} = \frac{1}{3} * \text{area of base} * \text{perpendicular height}$$
$$= \frac{1}{3} \pi r^2 h$$

If we cut out the circular base and cut along the slant length, we can flatten the cone out into its net - this produces a circle, and a sector of another circle:



$$\text{surface area of cone} = \text{area of base} + \text{area of curved surface}$$
$$= \pi r^2 + \pi r L$$

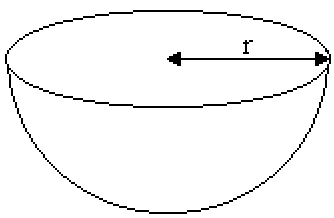
Spheres



$$\boxed{\text{surface area} = 4\pi r^2}$$

$$\boxed{\text{volume} = \frac{4}{3}\pi r^3}$$

If a sphere is cut in half, the volume is halved, but for the total area we need to now include the area of the circular cross-section:



$$\text{volume} = \frac{1}{2}\text{ a volume of complete sphere} = \frac{1}{2} * \frac{4}{3}\pi r^3$$

$$\boxed{\text{volume} = \frac{2}{3}\pi r^3}$$

$$\text{surface area} = \frac{1}{2}\text{ area of sphere} + \text{area of circular part} = \frac{1}{2} * 4\pi r^2 + \pi r^2$$

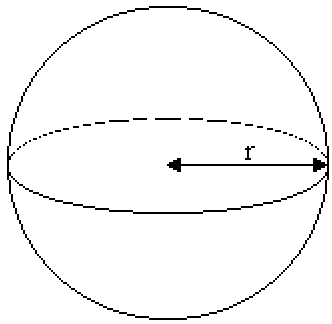
$$\boxed{\text{surface area} = 3\pi r^2}$$

Example

A golf ball has a diameter of 4.27cm. What is its surface area and volume? State the answers to one decimal place.

The equations for area and volume use radius, rather than diameter:

The radius, r = diameter/2 = 4.27/2 = 2.135cm



surface area $= 4\pi r^2 = 4*\pi*2.135^2 = 57.28 = 57.3\ cm^2$ (to 1 dp)

volume $= \frac{4}{3}\pi r^3 = \frac{4}{3}*\pi*2.135^3 = 40.765 = 40.8\ cm^3$ (to 1 dp)

DIMENSIONS AND FORMULAE - start of this chapter - contents

If we think of length as having 1 dimension, then area has 2 dimensions and volume has 3 dimensions:

quantity	number of dimensions	a possible unit
length	1	cm
area = one length*another length	2	cm^2
volume = one length*another length*another length	3	cm^3

Notice that the power of the unit is the same as the number of dimensions - either 1 , 2 or 3 (recall that cm is the same as cm^1).

We can study a formula, and from its dimensions, determine if it *could* represents a length, an area or a volume.

One way is to replace any variable standing for length by, say, cm:

- if the formula produces cm then it could be a formula for a length
- if the formula produces cm^2 then it could be a formula for an area

- if the formula produces cm^3 then it could be a formula for a volume

For example, the area of a circle is found from $A = \pi r^2$

If we replace r by cm, the unit of r^2 is cm^2, which is correct for an area.

Example

A student's notes say that the volume of a sphere is found from: $\text{volume} = \frac{4}{3} \pi r^2$

Could this formula be correct?

If we replace r by cm, we get $r^2 = cm^2$.

But the unit of volume is cm^3 , so the formula must be wrong.

(if you check back, you'll see that the r^2 in the student's formula should be r^3)

Example

Could the following be the formula for the surface area of something?

$$\text{surface area} = \pi r^2 + \pi r L^2 \quad (r = \text{radius}, L = \text{length})$$

If we replace r and L by cm:

- in the 1st term: r^2 has the unit cm^2
- in the 2nd term: $r L^2$ has the unit $cm*cm^2 = cm^3$

The first term has the unit of area, but the second term has the unit of volume, so the formula cannot represents an area.

Note: It is a general point that:

- all terms in *any* equation/formula must have the same units

Example

In the following, L, h and r all represent lengths. Which formula could *not* represent a volume?

1. 2Lhr
2. L(r + h)
3. Lr^3

Putting cm in place of each length:

1. The unit of 2Lhr = cm*cm*cm = cm^3 , so this could be a volume
2. The unit of L(r + h) = unit of Lr + Lh. Both terms have the unit cm*cm = cm^2 , which is an area unit, not a volume unit
3. The unit of Lr^3 is cm*cm^3 = cm^4 , which is not a volume unit

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Part 4: HANDLING DATA - contents

Statistics (Chapters 21 to 23)

Chapter 21

- TERMINOLOGY

- REPRESENTING DATA VISUALLY
- STATISTICAL SURVEYS

TERMINOLOGY - contents

- *Statistics* is concerned with collecting, analysing (processing) and representing data.

Data is a collection of observation. For example, a set of exam results is data. Information gathered about a group of people's favourite TV programs is data.

Raw data is a collection of facts which has not been processed in any way.

A *population* is everyone or everything in a sample being considered. If you were recording the ages of students in a class, all the students would be in the population. If you were considering the make of cars in a car park, all the cars would be in the population (so a populations does not always refer to people).

A *variable* is anything that can vary (be different) from one item to another. For example, ages of students in a classroom or the makes of cars in a car park are variables.

Qualitative variables are those such as the colour or make of a car, which are not numerical.

Quantitative variables are numerical - these may be discrete or continuous:

- *Discrete variables* are those which can only take certain particular values, but not in-between values

The number of children in a classroom might be, for example, 17 or 18, but not 17.4. So the number of children is a discrete variable.
Discrete variables are not necessarily whole numbers. For example, shoe sizes are discrete, but a size may be, for example, 8½, as well as 8 or 9 etc.

- *Continuous variables* are those which can take any value, usually between certain limits

  For example, between the shortest and tallest child in a class, the other children might have any height.
  Similarly, the age of a person might be anything from zero to over a hundred.

Data is often represented in ways which make it easier to see patterns in the data. Some of the following may be familiar from their use on TV and in newspapers etc.

REPRESENTING DATA VISUALLY - start of this chapter - contents

A student makes notes about 60 cars and their drivers as they enter a car park.

colour	men	women	total
blue	9	13	22
red	11	8	19
green	4	4	8
white	4	2	6
black	2	3	5

- The above is called a *frequency distribution* or a *frequency table* . It indicates the frequency of each variable (i.e. the number of times each occurs). For example, the frequency of men with blue cars is 9, while the frequency of women with green cars is 4, etc.

Data such as the above can be conveniently represented by a *bar chart* or a *pie chart* . These represent the variables and the frequencies in a single diagram.

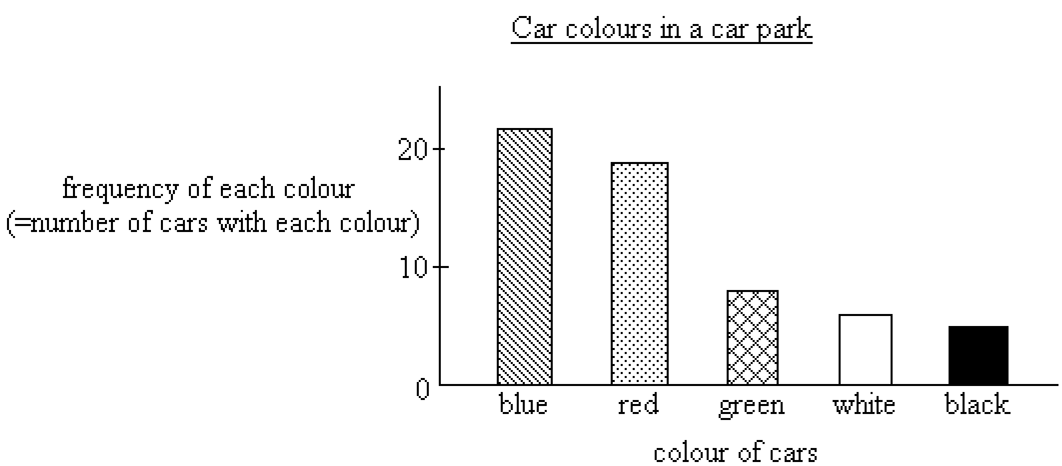
Bar charts

In a *bar chart* (or *bar graph*) we draw bars of equal width (sometimes we may just use lines).

There are various types of bar charts that can be used:

1. A vertical bar chart

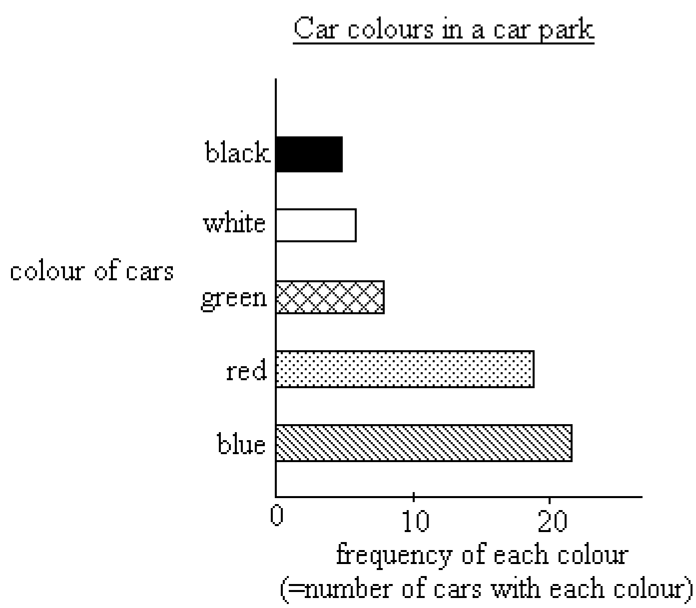
The following is a vertical bar chart, representing the total number of cars of each colour:



Notice that:

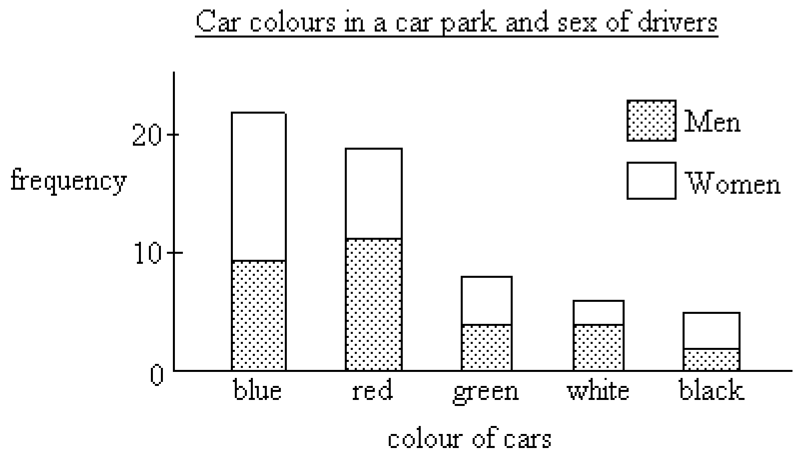
- the length of a bar is proportional to the frequency - if we add all the lengths we get the total number of cars, i.e. 60
- the bars or 'towers' do not touch
- there is no horizontal scale, just descriptions of the quantities (the variables)
- the graph is given a *title* (to describe it), and both the axes are labelled

2. A horizontal bar chart



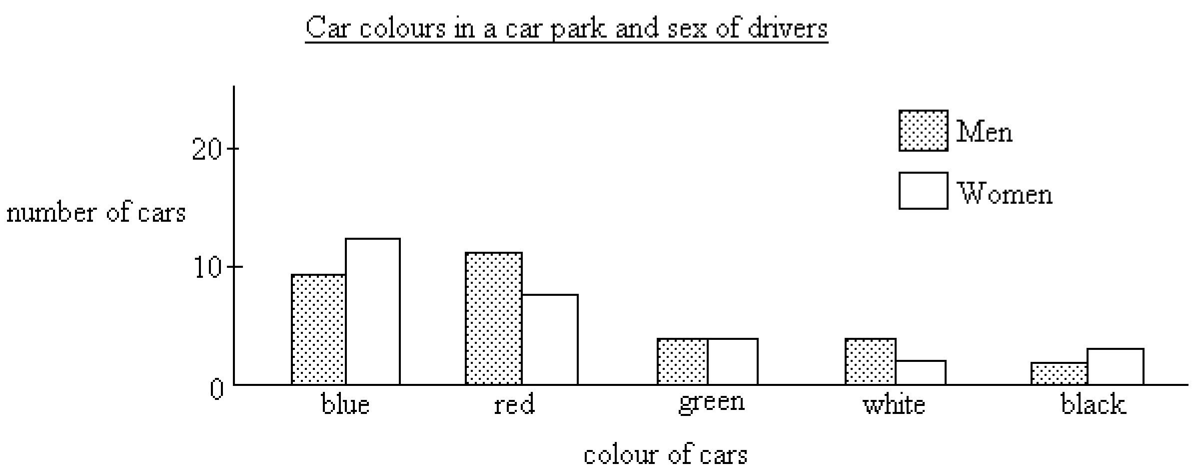
3. A proportionate bar chart (also called a component bar chart or a stacked bar chart)

This allows us to indicate the car colours and sex of drivers on the same chart:



4. A comparative bar chart

This is as the above, except that the bars representing numbers of men and women are placed side by side:



Pie charts

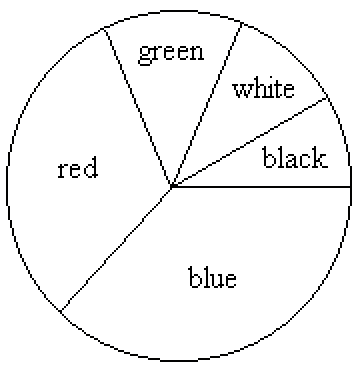
Recall the original data:

colour	men	women	total
blue	9	13	22
red	11	8	19
green	4	4	8
white	4	2	6
black	2	3	5

Here we will just consider the total number of each colour.

In the pie chart representation below, each segment of the 'pie' represents a car colour. The size of each segment represents the number of cars of each colour:

Car colours in a car park



Procedure :

A complete circle contains 360° , and to divide the pie into the appropriate proportions, we need to work out the angle corresponding to the number of cars of each colour.

There are 60 cars in total, so an angle of $\frac{360^\circ}{60}$ (= 6°) represents one car

So, for example, 6*22 = 132° represents all the blue cars.

Similarly, for the other colours, as idicated in the table below:

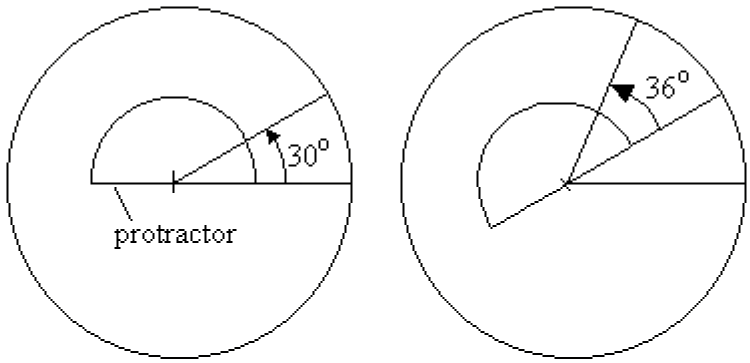
colour	number	angle (degrees)
blue	22	22*6 = 132°
red	19	19*6 = 114°
green	8	8*6 = 48°
white	6	6*6 = 36°
black	5	5*6 = 30°
	60	total = 360°

- It's worth adding up all the angles, to check that they come to 360° .

To draw the pie chart:

- draw a circle of a convenient size
- mark the centre
- draw a radius (anywhere, but it may be easiest to draw a horizontal radius)
- use a protractor to draw in the other radii
- label each sector appropriately

Assuming that the first radius has been drawn horizontally to the right, the following indicates how to place a protractor to mark the radius for the black cars, and then the radius relative to this for the white cars.

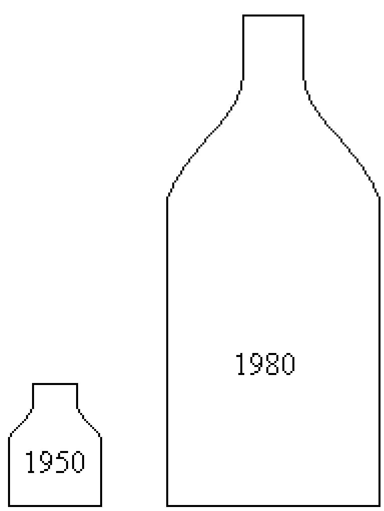


The process continues for all the other colours. When the first four sectors are complete, the 5th is automatically complete - you should measure it to see if it is 132° , as it should be.

Pictograms

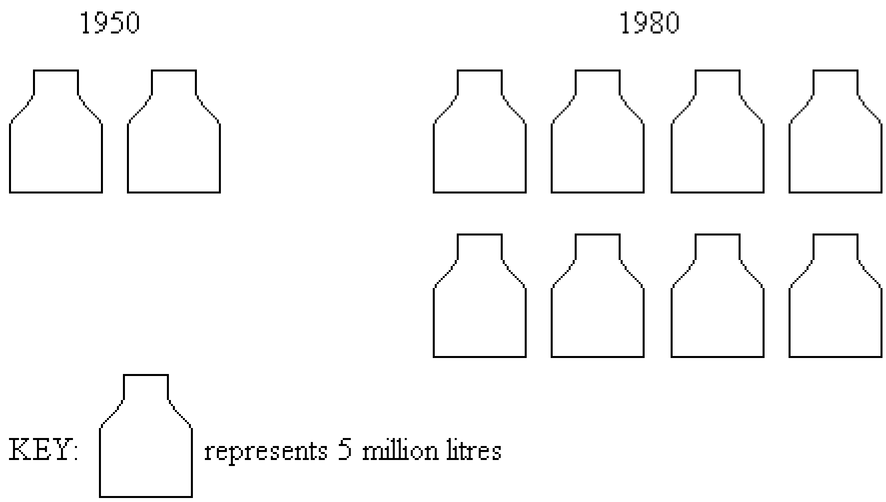
These are a pictoral way of representing data.

The following is meant to represent the increase in milk consumption in the UK between 1950 and 1980:



However, it is not clear whether we are meant to compare heights, widths or areas, each of which gives a different proportional increase.

The following is a way of making the information clearer:



Chronological graphs (or time-series graphs)

These are used to represent the change in one or more quantities over a time period:

- time is usually plotted along the horizontal axis

The following indicates the number of employees working for a company over a few years:

year	number of employees
1985	10
1986	15
1987	20
1988	40
1989	60

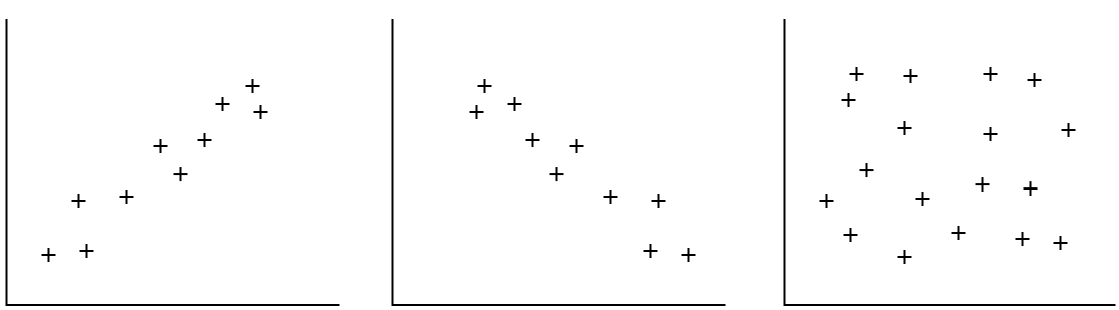
And the following is the data represented as a chronological graph (or chart):

Scatter diagrams

If we collect data about two different quantities, there may or may not be a relationship (called a 'correlation') between them.

We might expect a relationship between the age of a child, and his or her shoe size. But would probably not expect a relationship between the height of a student and their exam results.

The following are examples of *scatter diagrams* in which we simply plot one quantity against the other to see if the graphs reveals any apparent relationship:



positive correlation negative correlation no correlation

- A positive correlation (relationship) exists between two quantities if one increases when the other increases
- A negative correlation exists between two quantities if one decreases when the other increases
- If two quantities seem to be completely independent, we say that there is no correlation between them

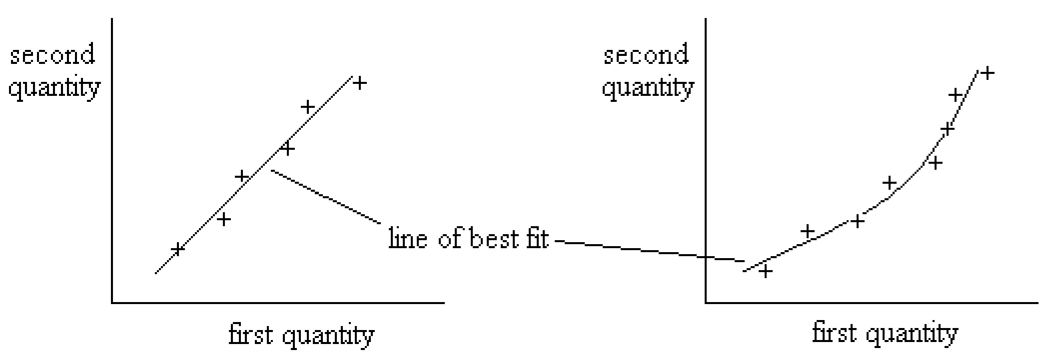
We would expect, for example, that the size of a child's shoe to increase with age, up to a certain limit, so these quantities have a positive correlation.

But for an adult, shoe size changes little, if at all, with age, so there is little or no correlation between the quantities.

Various terms are used to describe the amount of correlation:

- little or no correlation
- moderate positive correlation
- moderate negative correlation
- strong positive correlation
- strong negative correlation

It may be that the points on a scatter diagram seem to lie close to a line - in which case we can draw, by eye, *a line of best fit* . The following represent strong positive correlation, and a line of best fit has been drawn through each set of points:



STATISTICAL SURVEYS - start of this chapter - contents

These involve obtaining data from people by asking questions or using questionnaires. It may be that the relevant population (the parent population) is millions of people, in which case only a relatively few would be questioned. For this to be worthwhile, *those questioned must be representative of the whole population* .

For example, suppose that in order to obtain a prediction about the voting in an upcoming general election, the *sampling method* was to ask 10 men at football match in Rotherham how they planned to vote.

This would be a poor sampling method because:

- a very small sample of the voting population has been used
- they are all men
- they were not randomly selected from the voting population - they all happened to be at the same place at the same time
- they are all in one town

In short, the 10 men are not a representative sample of the voting population. To improve the method, we could:

- use a larger sample, say 1000
- ask men and women
- select them at random from those of voting age
- do the survey in a variety of locations, town and country

Sampling methods and bias

- *Random sampling* . In this case every member of a population has an equal chance of being chosen. One approach is to pick names at random from a list of the names in the relevant population
- *Stratified random sampling* . A population may contain groupings, and these groupings can be reflected in the way a sample of the population is chosen - this is better than just random sampling of the whole population. For example, if a population contains equal numbers of men and women, and a sample of 100 were required, then 50 of each would be randomly selected. The sample is then in the same proportion as the population.

A survey is said the be *biased* when the sample of the population selected is not representative of the whole population.

- An *advantage* of using random sampling or stratified random sampling is that it helps to avoid bias

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Part 4: HANDLING DATA - contents

Statistics (Chapters 21 to 23)

Chapter 22

- AVERAGES
- DISPERSION OF DATA

AVERAGES - contents

The following is raw data made up of results from a maths test:

5 7 6 6 4 9 8 3 5 2 4 6 5 6 1 3 2

When we state an average value of a set of data, we are representing all the data by a single number. In statistics, averages are thought of as 'measures of central tendency'. There are three different averages used, called *mean* , *median* and *mode* :

1. Mean (also called *arithmetic mean*)

- The mean of a set of values equals the sum of all of them divided by how many values there are, so:

$$\boxed{\text{mean} = \frac{\text{sum of all values}}{\text{number of values}}}$$

In the above example,

$$\text{mean} = \frac{82}{17} = 4.8$$

The mean value is what most people think of as being an average value - but the following two averages are also used in statistics.

2. Median

- The median of a set of values is the *data value* which occurs in the middle when the values are arranged in ascending (increasing) order

If the above test results are put in order we get:

1 2 2 3 3 4 4 5 5 5 6 6 6 6 7 8 9

There are 17 entries, and the middle one, the 9th, is 5, so the median is 5.

If there are an even number of entries, there is not a *single* middle entry - in this case the median equals the middle two data values added together and divided by 2. For example, for the data:

1 2 5 5 6 7 9 11 13 15

The median = (6+7)/2 = 6.5

3. Mode

- The mode is the *data value* which occurs most often (most frequently)

For the following set of values, the mode is 6:

1 2 2 3 3 4 4 5 5 5 6 6 6 6 7 8 9

Some sets of values do not have a mode. In the following, each value occurs once:

1 2 3 4 7 9 12

Some sets of values have more than one mode. For example, the following have the modes 4 and 7, since these both occur 3 times, which is more than any of the other values:

1 1 2 2 3 4 4 4 5 6 6 7 7 7 8 9 9

Memory jogger : The following may help you avoid mixing up mode with the other two averages:

- think of **MODE** as standing for **M** ost **O** ften **D** ata **E** ntry

DISPERSION OF DATA (Note: 'to disperse' means 'to spread out') - start of this chapter - contents

Sets of data may be compared using their mean, median and mode. However:

- a *disadvantage* of using an average value to represent a set of data is that it does not tell you how spread out (disperse) the data is

For example, the following have the same mean value, but the data entries are spread out very differently about the mean:

- the mean of 102 and 104 = (102 + 104)/2 = 206/2 = 103
- the mean of 2 and 204 = (2 + 204)/2 = 206/2 = 103

Hence, in addition to the idea of averages, in statistics we define quantities which provide us with measures of spread or dispersion of data.

1. Range

This measures spread simply as how far apart the biggest and smallest items of data are:

- range = maximum value - minimum value

In the set of data: 1 2 6 7 9 13 15, the range = 15 -1 = 14

- One *disadvantage* to using range as a measure of spread, is that it depends only on the extreme values, i.e. the highest and lowest data values, and tells you nothing about the in-between values

2. Interquartile range

Before defining interquartile range, we need to define the term 'quartile':

- the quartiles of a set of data are those *data values* which divide the data into four equal parts, i.e. into four quarters, after the data has been arranged in numerical order

The second quartile is the same as the median, since the median divides the data into 2 halves, and a half is the same as two quarters.

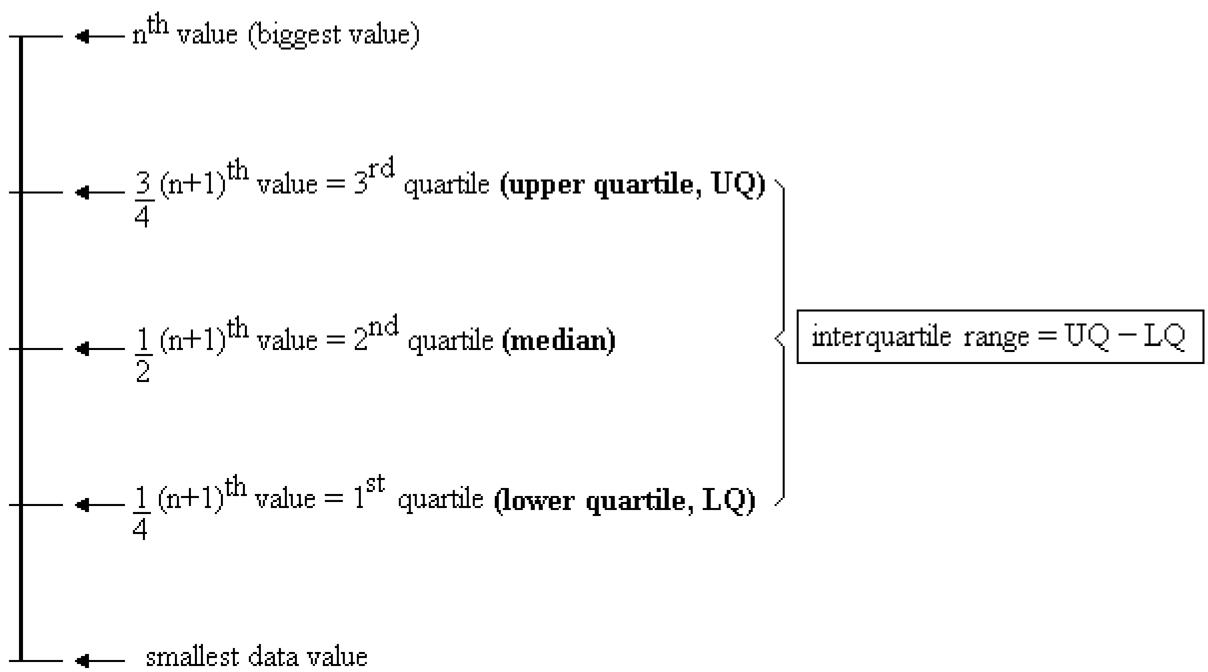
The following represents15 pieces of data, arranged in increasing size from 5 to 24. The diagram indicates the quartiles and what we mean by the term 'interquartile range':

24
23
23
20 ← 12^{th} value = 3^{rd} quartile (called upper quartile, UQ)
18
18
18
15 ← 8^{th} value = 2^{nd} quartile (same as median)
14
13
11
10 ← 4^{th} value = 1^{st} quartile (called lower quartile, LQ)
9
8
5

$$\boxed{\text{interquartile range} = UQ - LQ}$$
$$= 20 - 10$$
$$= \underline{10}$$

The quartiles occur at the 4^{th} , 8^{th} , and 12^{th} positions out of the 15 pieces of data.

To be more general, if we let the letter **n** stand for any number of pieces of data then, we can work out the quartiles according to:



To relate this to the previous data, replace n by 15, and see that the 1^{st} quartile is the $\frac{1}{4}(15+1)^{th} = 4^{th}$ value etc.

If n is fairly large, then adding 1 to it does not make much difference, so the first quartile value is the $\frac{1}{4}\ n^{th}$ value rather than the $\frac{1}{4}(n+1)^{th}$ value etc.

What if the quartiles do not coincide with particular items of data? Consider the data:

1 2 5 5 6 7 9 11 13 15

For these the 2^{nd} quartile (the median) is the $\frac{1}{2}(10+1)^{th}$ value = the $5\frac{1}{2}^{th}$ value. But what does the 'fifth and a halfth' value mean? We interpret this as being between the 5^{th} and 6^{th} value, so the 2^{nd} quartile is the mean of these:

- second quartile (= median) = ½ (6+7) = 6.5

Note that:

- An *advantage* of using interquartile range is that it provides a better measure of spread than ordinary range, because it does not depend on the extreme values of the data

3. Standard deviation

Note - you should check with your teacher or your syllabus which equations you will be given in the exams - you would probably be given the next one if you needed it.

The following is the definition of standard deviation (it is provided in the examination) - it contains some symbols not used so far:

$$\boxed{\text{standard deviation, } s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}}$$

- x = the value of any item if data, and
- $\bar{x}$ = the mean value of all the data = $\dfrac{\text{sum of all data values}}{\text{number of values}}$

said as 'x-bar'

The symbol Σ is a Greek letter said as 'sigma' – it is equivalent to our letter S.

We use Σ as an instruction to add together (i.e. to Sum), whatever quantity follows it.

So, for example, we can write: $\boxed{\bar{x} = \dfrac{\Sigma x}{n}}$ (n = number of data items)

- The quantity in the bracket $(x - \bar{x})$ is called the 'deviation from the mean' of x.

The quantity inside the square root sign is called the 'variance', i.e.

$$\text{variance} = \frac{\Sigma (x - \bar{x})^2}{n}$$

Example

The following are points scored in 11 games of rugby:

13, 23, 45, 11, 56, 32, 75, 17, 29, 62, 19

Calculate the standard deviation of the scores.

We set up a table to list the above scores and then work out:

- the mean value of the scores
- the deviation of each score from the mean
- the squares of the deviations

x	deviation $x - \bar{x}$	squared deviation $(x - \bar{x})^2$
13	-21.73	472.2
23	-11.73	137.6
45	10.27	105.5
11	-23.73	563.1
56	21.27	452.4
32	-2.73	7.5
75	40.27	1621.7
17	-17.73	314.4
29	-5.73	32.8
62	27.27	743.7
19	-15.73	247.4
$\Sigma x = 382$		$\Sigma (x - \bar{x})^2 = 4698.3$
$\bar{x} = \frac{\Sigma x}{n} = \frac{382}{11} = 34.73$		

Now we can complete the question:

$$\text{variance} = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{4698.3}{11} = 427.1$$

$$\text{standard deviation, } s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

$$= \sqrt{427.1}$$

$$\underline{s = 20.7}$$

Note:

- Remember that standard deviation is a measure of spread (deviation) - the more spread out the values are from the mean value, the greater the standard deviation
- An *advantage* of using the standard deviation is that it includes *every* item of data (unlike the range or the interquartile range)

GCSE Maths Notes

Part 4: HANDLING DATA - contents

Statistics (Chapters 21 to 23)

Chapter 23

- FREQUENCY DISTRIBUTIONS
- GROUPED FREQUENCY DISTRIBUTIONS

FREQUENCY DISTRIBUTIONS - contents

As previously mentioned:

- the *frequency* of a item of data is the number of times that it occurs, and
- by counting the frequency of each item of data we obtain a *frequency distribution* or a *frequency table*

The following are exam marks out of 10 for 36 students:

5	7	6	6	4	9	8	3	5
6	6	4	9	8	3	5	2	4
2	4	6	5	3	1	3	2	6
6	5	6	1	5	3	3	2	9

The above is raw data. One way to produce a frequency distribution from the raw data is to use a *tally chart* . In the left-hand column we list each different value that occurs in the data - we see that numbers from 1 to 9 occur. Then, in the tally column, we put a 1 for each time a value occurs. Usually, when we reach 5, the fifth entry is indicated by putting a line through a group of four:

mark	tally	frequency (=number with a given mark)
1	11	2
2	1111	4
3	~~1111~~ 1	6
4	1111	4
5	~~1111~~ 1	6
6	~~1111~~ 111	8
7	1	1
8	11	2
9	111	3
		total = 36

To complete the tally column, you could simply count all the 1s in the data, then all the 2s etc. However, it is easy to miss an item doing this. It is usually better to work though the data in order, crossing off each item as it is used - so you could, for example:

- start top left in the data, working along each row in turn
- put a 1 against each mark in the table as that mark occurs in the data
- cross off each mark of data when it has been recorded in the table

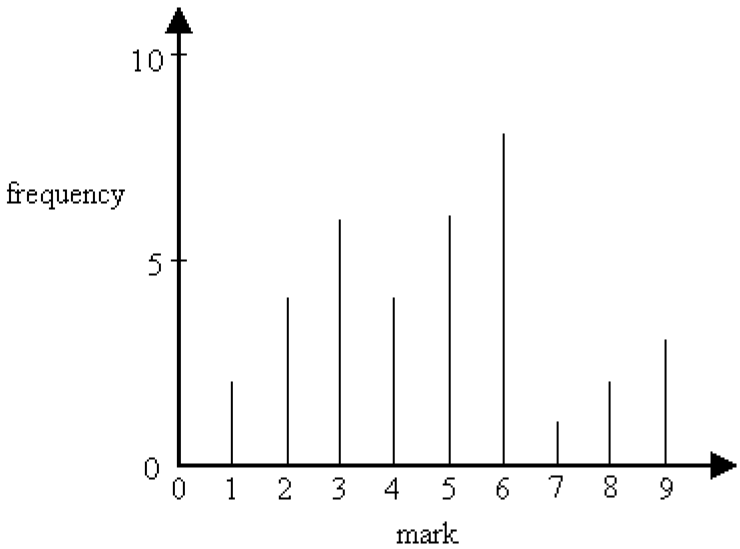
Since the frequency column contains the number of times each mark occurs, the total frequency must equal the total number of marks. You should add up the total frequency as a check - if it does not equal the total number of items in the data, then you've missed something or counted something more than once.

Mode of a frequency distribution

MODE = Most Often Data Entry - so it is the data item that occurs most frequently. We can see directly from the frequency distribution table that the mode of the data is 6, which occurs 8 times.

We could also draw a *line graph* (or *line chart*), which is a bar graph in which we use lines in place of bars, and see from this that the mode is 6:

Line graph representing marks of students in a test



Mean of a frequency distribution

We can work out the mean of the above set of marks in the normal way, i.e. by adding them all together and dividing by how many there are:

$$\text{mean} = \frac{\text{sum of all values}}{\text{number of values}} = \frac{172}{36} = 4.8$$

But we can also do this by adding an extra column to the tally chart:

mark	tally	frequency (=number with a given mark)	frequency * mark
1	11	2	2*1 = 2
2	1111	4	4*2 = 8
3	~~1111~~ 1	6	6*3 = 18
4	1111	4	4*4 = 16
5	~~1111~~ 1	6	6*5 = 30
6	~~1111~~ 111	8	8*6 = 48
7	1	1	1*7 = 7
8	11	2	2*8 = 16
9	111	3	3*9 = 27
		total = 36	total = 172

From the two totals, we get the mean = 172/36 = 4.8, as before.

We can represent the above procedure in a shorthand notation commonly used in statistics:

The symbol Σ is a Greek letter said as 'sigma' – it is equivalent to our letter S.

We use Σ as an instruction to add together (i.e. to Sum), whatever quantity follows it.

If we let letter f stand for the frequencies in the above table, then:

- the symbol Σf (said as 'sigma f') means: 'add together all the values of f '

Also, if we use x to stand for the marks, then:

- the symbol Σfx ('sigma fx') means: 'add together all the values of f*x'

Adding these symbols to the above table, gives us:

mark (x)	tally	frequency (f)	fx
1	11	2	2*1 = 2
2	1111	4	4*2 = 8
3	~~1111~~ 1	6	6*3 = 18
4	1111	4	4*4 = 16
5	~~1111~~ 1	6	6*5 = 30
6	~~1111~~ 111	8	8*6 = 48
7	1	1	1*7 = 7
8	11	2	2*8 = 16
9	111	3	3*9 = 27
		$\Sigma f = 36$	$\Sigma fx = 172$

$$\boxed{\text{mean} = \frac{\Sigma fx}{\Sigma f}}$$

$$= \frac{172}{36} = 4.8$$

Remember that the Σs are intructions to add together what follows them.
So, though we have a Σ on the top and bottom in the equation, they do not cancel each other.

Median of a frequency distribution

Arranging the original data in order we get:

1 1 2 2 2 2 3 3 3 3 3 3 4 4 4 4 5 5 5 5 5 5 6 6 6 6 6 6 6 6 7 8 8 9 9 9

The median is the middle value. There are an even number (36) of values so we calculate the mean of the middle two, the 18^{th} and 19^{th} . These are both 5, so their mean is 5, which is the median value.

We can also find the median by producing a *cumulative frequency distribution* from the data. We do this by first of all forming a table. The entry in the cumulative frequency column is the total number of marks less than a specified value - compare the next table with the previous tally chart to see how the cumulative frequencies have been worked out:

mark	cumulative frequency
less than 2	2
less than 3	$2 + 4 = 6$
less than 4	$6 + 6 = 12$
less than 5	$12 + 4 = 16$
less than 6	$16 + 6 = 22$
less than 7	$22 + 8 = 30$
less than 8	$30 + 1 = 31$
less than 9	$31 + 2 = 33$
less than 10	$33 + 3 = 36$

Or, using the 'less than' symbol '<' →

mark	cumulative frequency
< 2	2
< 3	6
< 4	12
< 5	16
< 6	22
< 7	30
< 8	31
< 9	33
< 10	36

The above is a 'less than' table. Sometimes a 'less than or equal table' may be used, in which the entry in the cumulative frequency is the total number of marks less than or equal to a specified value.

Notice that the final cumulative frequency, 36, is the total number of items of data.

Since there are 36 entries, the median is the average of the 18^{th} and 19^{th} values:

1. these do not fall in the 'less than 5' class, as this only has 16 members, so they must both be 5 or higher
2. they both fall in the class 'less than 6', since this has 22 members, so they must both be less than 6

Since the 18th and 19th values are both 5 or higher [from (1)], and both less than 6 [from (2)], then they must both be 5. So, the median is the average of 5 and 5, which is 5.

GROUPED FREQUENCY DISTRIBUTIONS - start of this chapter - contents

Showing the frequencies for individual data items, as for the marks in the above case, is fine when the number of different items is quite small. However, if there are lots of different items it is found to be more useful to group the items of data into classes, and count how many items are in each class - this is called a grouped frequency distribution.

- A *grouped frequency distribution* indicates how many times items occur in each specified *class*

The following is the time to the nearest minute that 36 students took to complete an examination:

21	10	24	14	34	22
13	22	26	23	24	32
24	15	21	27	22	19
25	16	31	18	27	16
11	25	23	22	24	29
28	21	17	26	12	23

This time, we put a 1 in the tally column when a time falls in the specified range of values:

time taken (to nearest minute)	tally	frequency (=number of students)
10-14	~~1111~~	5
15-19	~~1111~~ 1	6
20-24	~~1111~~ ~~1111~~ 1111	14
25-29	~~1111~~ 111	8
30-34	111	3
		total 36

Some terminology:

- The data has been grouped into five *classes* or *class intervals*
- *Class boundaries* are the boundaries (or divides) between classes. Classes cannot overlap, or else an item of data might fall in two different classes. The times were recorded to the nearest minutes, so:

    - The class boundaries of the 10-14 class are 9.5 and 14.5 minutes - all those times greater than or equal to 9.5 minutes but less than 14.5 minutes fall in the 10-14 class
    - The class boundaries of the 15-19 class are 14.5 and 19.5 minutes - all those times greater than or equal to 14.5 minutes but less than 19.5 minutes fall in the 15-19 class
    - Hence, the class boundary between the 10-14 and 15-19 class is 14.5 minutes, since this is the divide between them
    - The other class boundaries are 19.5, 24.5, 29.5 and 34.5 minutes

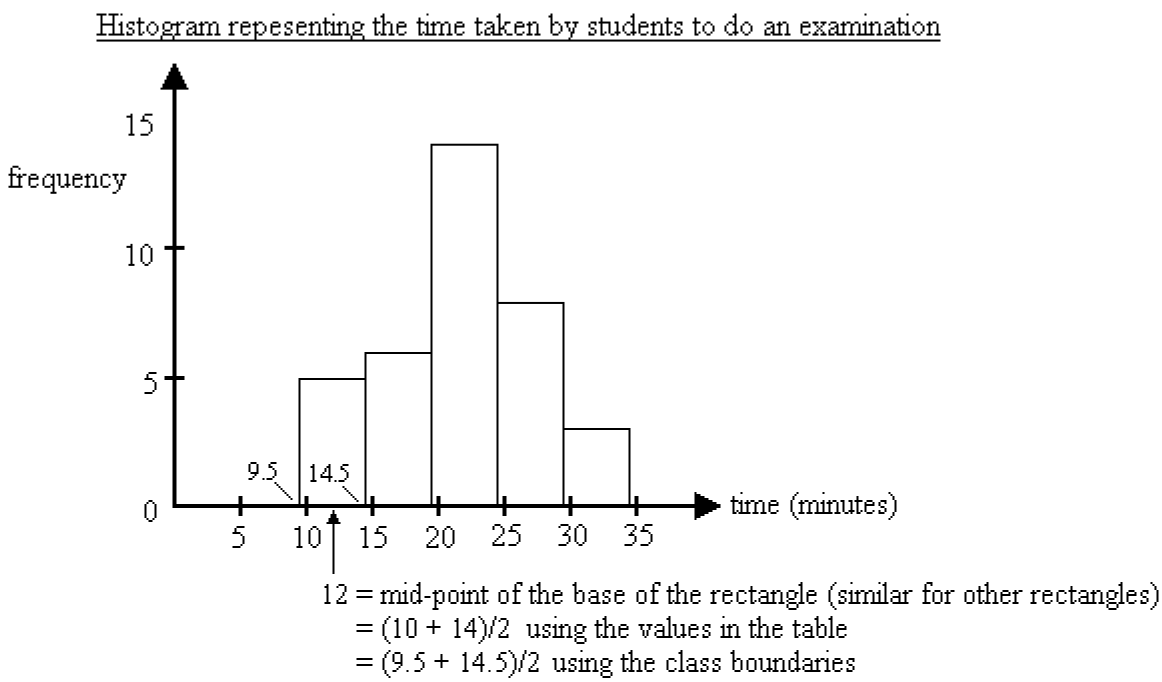
- A *class width* is the difference between the lower and upper class boundaries of a class. The class width is the same for all the above classes, for example, 14.5 - 9.5 = 5 minutes

Histograms

A *histogram* is like a bar chart, except:

- the width of a rectangle now represents the class width
- there are no gaps put between the classes, the rectangles touch each other along their sides
- there are numbers along both axes

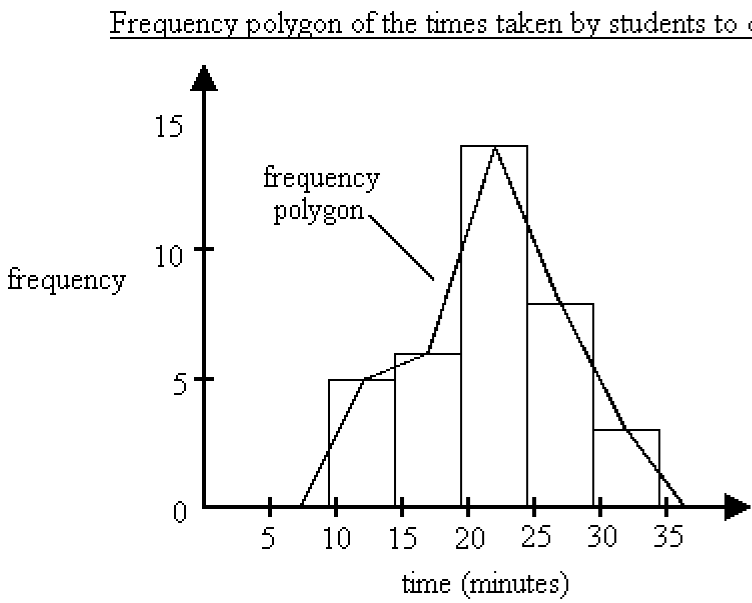
The following histogram represents the above grouped frequency distribution:



To construct the histogram:

- mark the horizontal and vertical axes as shown with the maximum values and suitable scales
- mark the class boundaries at 9.5, 14.5 etc. - since theses are where the vertical lines are to be drawn
- draw rectangles of appropriate height at each pair of class boundaries
- label both axes and give the histogram a title

An alternative representation to the histogram is a **frequency polygon** - this is produced from a histogram by joining the mid-points of the top of the rectangles by *straight lines* :



Notice that to complete the polygon, we:

- join the middle of the first bar to middle of where the previous bar would be, at the point where frequency = zero
- join the middle of the last bar to middle of where the next bar would be, at the point where frequency = zero

It is not be essential to draw the histogram in order to draw the polygon, but drawing the histogram does help in correctly locating the points to be joined to form the polygon.

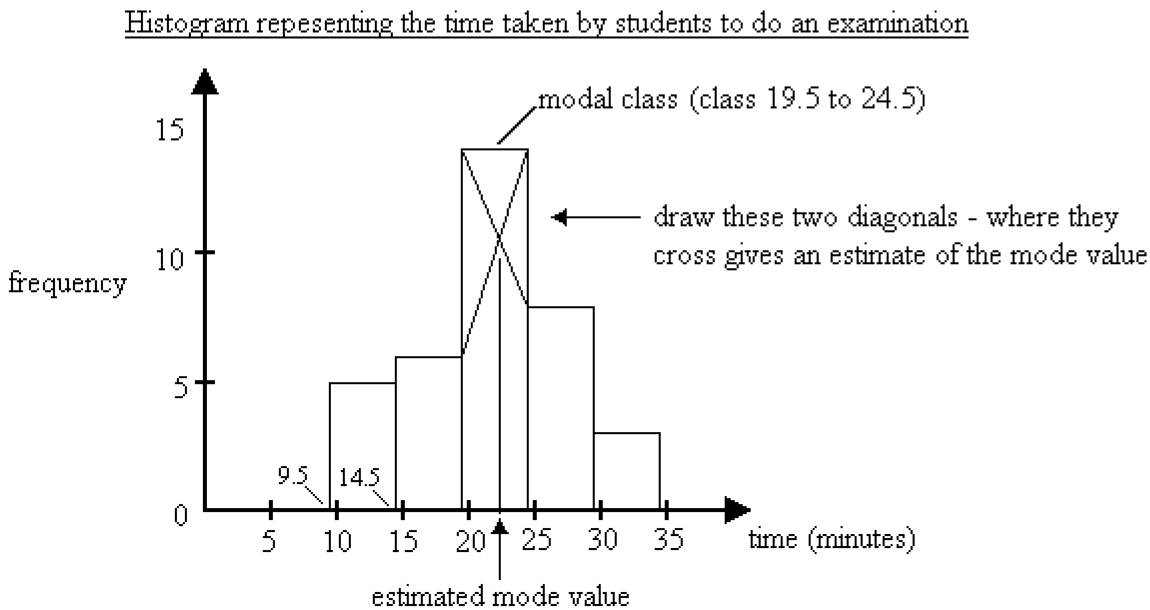
Mode and modal class of a grouped frequency distribution

The mode of a set of data is the value that occurs most often, i.e. has the highest frequency.

In a grouped frequency distribution, we do not have individual frequencies shown on the histogram, but rather we have the frequencies for each class. However, we define:

- the *modal class* of a distribution is the class with the highest frequency

The modal class for the above grouped frequency distribution is indicated below, together with a method of estimating the value of the mode from the histogram:



Mean of a grouped frequency distribution

We can find the mean in a similar way to previously for a frequency distribution. However, for the x values, we now use the mid-points of the various classes, as indicated below:

time taken (to nearest minute)	x (mid-point of class)	frequency (=number of students)	fx
10-14	12	5	60
15-19	17	6	102
20-24	22	14	308
25-29	27	8	216
30-34	32	3	96
		$\Sigma f = 36$	$\Sigma fx = 782$

$$\text{mean} = \frac{\Sigma fx}{\Sigma f} = \frac{782}{36} = 21.7$$

The above method of finding the mean is a bit of an approximation, as we can see if we work out the mean in the normal way:

$$\text{mean} = \frac{\text{sum of all values}}{\text{number of values}} = \frac{787}{36} = 21.9$$

Median of a grouped frequency distribution

Placing the original data in numerical order, we get:

10 11 12 13 14 15 16 16 17 18 19 21 21 21 22 22 22 22
23 23 23 24 24 24 24 25 25 26 26 27 27 28 29 31 32 34

The median (2nd quartile) = the ½ (36+1)th value = 18½ value, i.e. the average of the 18th and 19th entry, which is (22+23)/2 = 22.5

We can also get an estimate of the median from a cumulative frequency distribution table for the data.

Here we use 14.5 etc. as the divisions between the classes, and then add up the total number of times below each division:

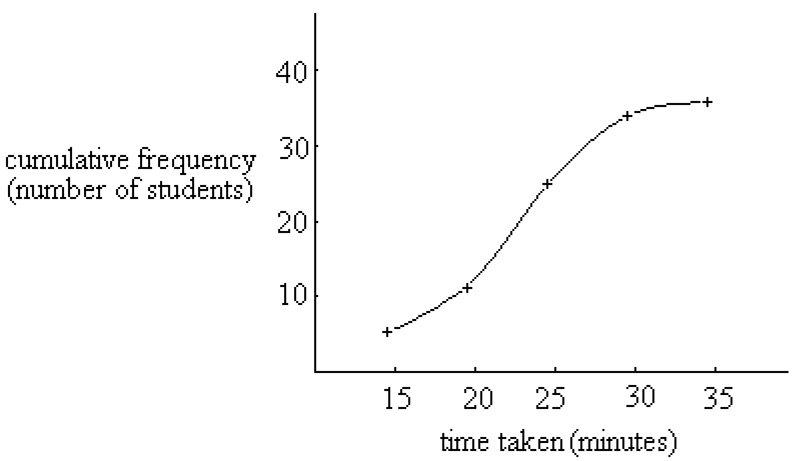
time taken (minutes)	cumulative frequency (number of students)
less than 14.5	5
less than 19.5	11
less than 24.5	25
less than 29.5	33
less than 34.5	36

The table tells us that the 18th and 19th members fall somewhere in the class 'less than 24.5', but does not give a value for median as specific as the previous method

We can get a better estimate for the median by plotting a *cumulative frequency graph/curve* :

Cumulative frequency graphs

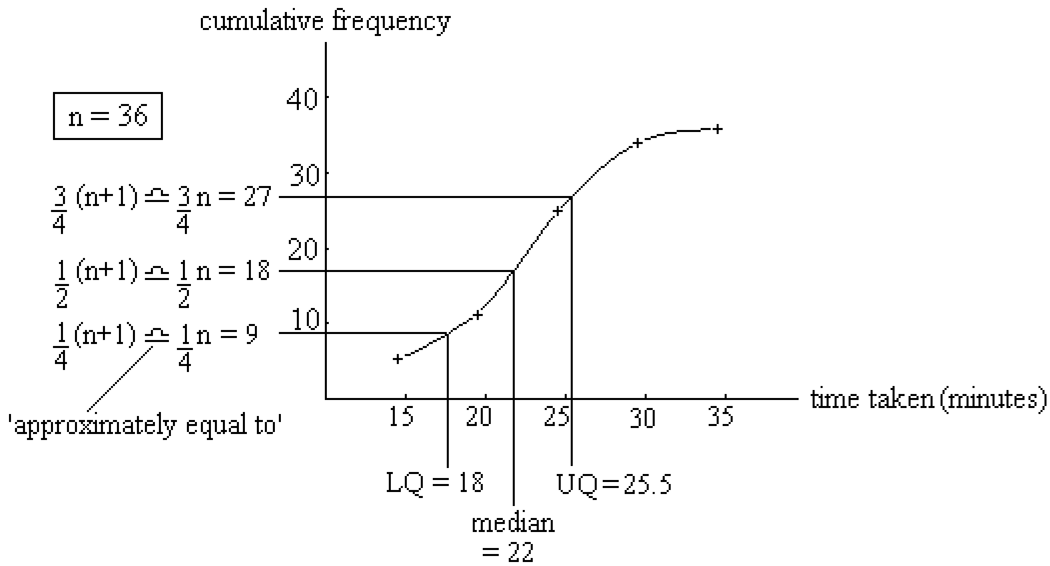
- a *cumulative frequency curve* is produced by plotting the points from the above table, and joining the points to form a smooth curve:



Cumulative frequency graphs are shaped like a stretched S, and are called 'ogives' (a term taken from architecture).

An advantage of a cumulative frequency curve is that we can get estimates of the 1st quartile (or lower quartile, LQ), the 2nd quartile (median) and 3rd quartile (or upper quartile, UQ) directly from the graph.

Recall that the 1st quartile is the ¼(n+1)th value. Since n=36, which is reasonably large compared to 1, we can use the approximations indicated below:



Recall that interquartile range = upper quartile - lower quartile = 25.5 - 18 = 7.5 minutes

Frequency density

Rather than draw a histogram of frequency (upwards) against class interval, some histograms are drawn with 'frequency density' on the vertical scale - definition:

$$\boxed{\text{frequency density} = \frac{\text{frequency of class interval}}{\text{width of class interval}}}$$

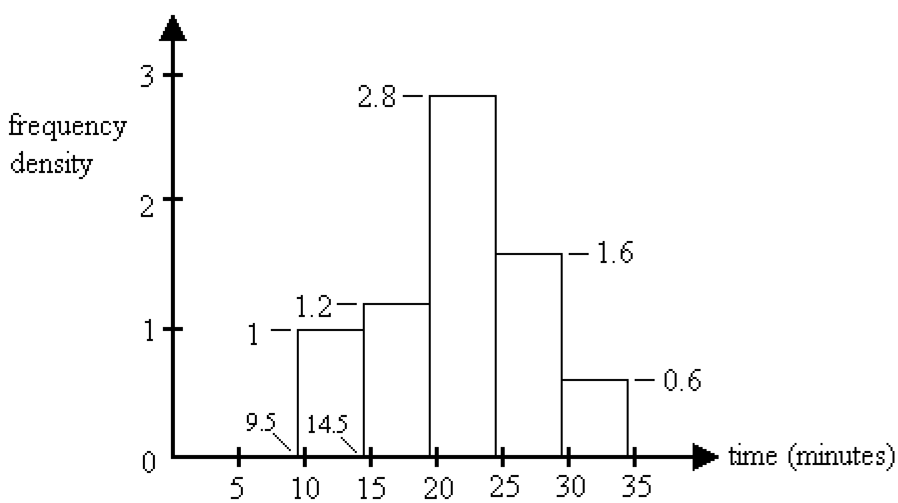
In the data used for the previous histogram, the class interval was 5 for all the classes (e.g. 14.5 - 9.5 etc.). Applying the above definition to that data, we divide all the frequencies by 5 to get:

time taken (to nearest minute)	frequency (=number of students)	frequency density = $\frac{\text{frequency of class interval}}{\text{width of class interval}}$
10-14	5	1
15-19	6	1.2
20-24	14	2.8
25-29	8	1.6
30-34	3	0.6

Note - Physical density is '*mass per unit volume* ', for example, grams per cm^3 . We use the term 'frequency density' in a similar way - it is the '*frequency per unit class interval* '.

Drawing the histogram:

Histogram repesenting the time taken by students to do an examination



Given a histogram such as the above, we can work out the frequency of any particular class interval by reversing the calculation for frequency density:

Since, $$\text{frequency density} = \frac{\text{frequency of class interval}}{\text{width of class interval}}$$

Then, $$\boxed{\text{frequency of class interval} = \text{frequency density} * \text{width of class interval}}$$

For example, in the third interval:

$$\text{Frequency} = \underline{2.8 * 5 = 14}$$

Notes:

In the above type of histogram, *with frequency density on the vertical axis* :

- the *area* of a column equals frequency (= frequency density * width of class interval) - this is so even if the columns have different class intervals
- the sum of the areas of all the columns equals the total frequency

In the previous type of histogram, *with frequency on the vertical axis* :

- the *height* of a column represents frequency
- the sum of the heights of all the columns equals the total frequency

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GCSE Maths Notes

Part 4: HANDLING DATA - contents

Probability

Chapter 24

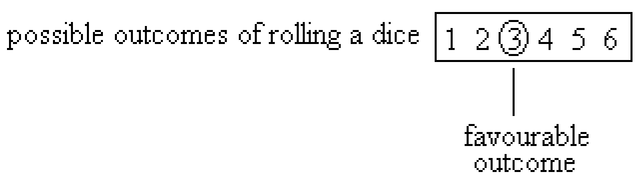
- BASIC IDEAS
- RANGE OF PROBABILITIES
- TOTAL PROBABILITY
- MUTUALLY EXCLUSIVE EVENTS - THE ADDITION LAW
- INDEPENDENT EVENTS - THE MULTIPLICATION LAW
- PROBABILITY TREE
- EXPERIMENTAL PROBABILITY
- CONDITIONAL PROBABILITY

BASIC IDEAS - contents

One roll of a dice or a toss of a coin is called a *trial* , and the *outcome* of a trial is called an *event* .

When a dice is rolled, there are six possible outcomes in a single trial - the numbers from 1 to 6.

If we are interested in the *chance* or *probability* of getting a 3, for example, then the 3 is called the *favourable outcome* .



We define the *probability* of a particular event occurring in a trial as:

$$\boxed{\text{Probability} = \frac{\text{number of possible favourable outcomes}}{\text{total number of possible outcomes}}}$$

We usually let P = probability, so for example, P(3) means the probability of getting a 3 in a single roll of a dice:

$$P(3) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{1}{6}$$

We describe this as a probability of 'one in six'.

Note - as in the above case, it sometimes helps to list all possible outcomes and then circle the favourable ones - then the above definition can be used directly.

Example

What is the probability of getting a head when a coin is tossed once?

possible outcomes of tossing a coin $\boxed{H \; T}$

favourable outcome

$$P(\text{head}) = \frac{\text{favourable outcomes}}{\text{possible outomes}} = \frac{1}{2}$$

Probabilities can be expressed a fractions, decimals or percentages, so:

P(head) = 1/2 = 0.5 = 50%.

Example

What is the probability of getting the jack of hearts when one card is drawn from a pack of cards?

A pack of cards has 4 suits, with 13 cards in each suit (1 to 10, plus jack, queen, king):

clubs	1 2 3 4 5 6 7 8 9 10 J Q K
diamonds	1 2 3 4 5 6 7 8 9 10 J Q K
hearts	1 2 3 4 5 6 7 8 9 10 J Q K
spades	1 2 3 4 5 6 7 8 9 10 J Q K

$$P(\text{jack of hearts}) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{1}{52}$$

Example

What's the probability of getting a jack of any suit when one card is drawn from a pack?

clubs	1 2 3 4 5 6 7 8 9 10 J Q K
diamonds	1 2 3 4 5 6 7 8 9 10 J Q K
hearts	1 2 3 4 5 6 7 8 9 10 J Q K
spades	1 2 3 4 5 6 7 8 9 10 J Q K

This time there are 4 favourable outcomes, i.e. any of the jacks in the pack, so:

$$P(\text{jack}) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{4}{52} = \frac{1}{13}$$

RANGE OF PROBABILITIES - start of this chapter - contents

What is the probability of getting a head or a tail when a coin is tossed?

Clearly one of these is <u>bound</u> to occur:

possible outcomes of tossing a coin $\boxed{H\ T}$

favourable outcomes

According the the definition of probability:

$$P(\text{head or tail}) = \frac{\text{favourable outomes}}{\text{possible outcomes}} = \frac{2}{2} = 1$$

- Thus, something that is *certain* to occur has a probability of one (or, equivalently, 100%)

What is the probability of getting a 7 in a single roll of a dice?

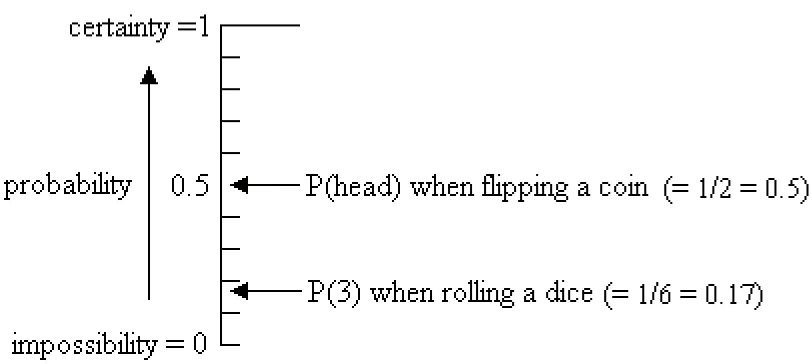
Clearly, a 7 <u>cannot</u> occur, and according to the definition of probability:

$$P(7) = \frac{\text{favourable outomes}}{\text{possible outcomes}} = \frac{0}{6} = 0 \quad \text{(Note: zero divided by anything = zero)}$$

- Thus, something that *cannot* occur has a probability of zero

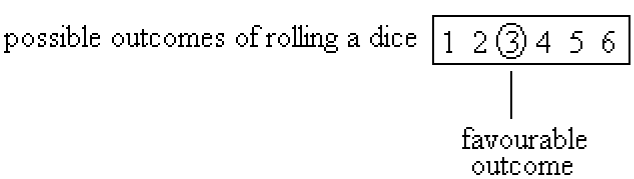
So, if a calculation produces a probability greater than 1 or less than zero (negative) then a mistake has been made somewhere.

We can represent the range of probabilities as:



TOTAL PROBABILITY - start of this chapter - contents

As already seen, when a dice is rolled, P(3) = 1/6.



What is the probability of *not* getting a 3?

possible outcomes of rolling a dice $\boxed{①②3④⑤⑥}$

favourable outcomes

If we do *not* get a 3 , then we *must* get a 1 or 2 or 4 or 5 or 6, so:

$$P(\text{not a }3) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{5}{6}$$

Observe that: $P(3) + P(\text{not a }3) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$

Getting a 3 when rolling a dice or not getting a 3 are called *complementary events* - if one does not occur then the other must. Hence, the probability of one or the other occurring is 1, i.e. certainty.

So, for complementary events:

$$\boxed{\text{the probability of an event occurring} + \text{the probability of it }\underline{\text{not}}\text{ occurring} = 1}$$

(Note - there is nothing special about using 3 in the above - any number on the dice could be used)

Example

What is the probability of *not* drawing a heart when a card is drawn from a pack?

Drawing a heart or not drawing a heart are complementary event - one or the other *must* occur, so:

P(heart) + P(not a heart) = 1

So, P(not a heart) = 1 - P(heart)

$$P(\text{heart}) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{13}{52} = \frac{1}{4} = 0.25 = 25\%$$

So, P(not a heart) = 1 - P(heart) = 1 - 0.25 = 0.75 = 75%

MUTUALLY EXCLUSIVE EVENTS - THE ADDITION LAW - start of this chapter - contents

What is the probability of getting a 3 or a 5 in a single roll of a dice?

possible outcomes of rolling a dice $\boxed{1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6}$

favourable outcomes

$$\text{So, } P(3 \text{ or } 5) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{2}{6} = \frac{1}{3}$$

$$\text{Now, the } \underline{\text{individual}} \text{ probabilities, } P(3) = \frac{1}{6} \text{ and } P(5) = \frac{1}{6}$$

$$\text{Notice that } P(3 \text{ or } 5) = \frac{1}{3} = \frac{1}{6} + \frac{1}{6} = P(3) + P(5)$$

Now, rolling a dice cannot produce *both* a 3 *and* a 5 at the same time. Similarly, tossing a coin cannot produce *both* a head *and* a tail at the same time. These are said to be *mutually exclusive events* :

- Mutually exclusive events are those which cannot occur together - if one event occurs, the other(s) cannot

We see in the above calculation, that to find the probability of a 3 OR a 5 we ADD the probability of a 3 to the probability of a 5. This is a general rule for mutually exclusive events:

- If the individual probabilities of several *mutually exclusive events* are P_1 , P_2 , P_3 , ... etc., then the probability P of one or other occurring is given by *the addition law of probability* :

$$\boxed{P = P_1 + P_2 + P_3 + \ldots\ldots}$$

Example

A card is drawn from a pack of cards. What is the probability that the card will be a 4 or the king of hearts?

We can firstly do this as before, by listing all possible outcomes, and counting the favourable ones:

clubs	1 2 3 4 5 6 7 8 9 10 J Q K
diamonds	1 2 3 4 5 6 7 8 9 10 J Q K
hearts	1 2 3 4 5 6 7 8 9 10 J Q K
spades	1 2 3 4 5 6 7 8 9 10 J Q K

$$P(\text{a 4 or king of hearts}) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{5}{52}$$

Using the addition law we get:

$$P(\text{a 4 or king of hearts}) = P(4) + P(\text{ king of hearts})$$
$$= \frac{4}{52} + \frac{1}{52} = \frac{5}{52}$$

Memory jogger :

- **MEAL = M** utually **E** xclusive - **A** ddition **L** aw

The above may help you avoid mixing up the addition law with the following law.

INDEPENDENT EVENTS - THE MULTIPLICATION LAW - start of this chapter - contents

What is the probability getting a head when a coin is tossed, *and* getting a 4 when a dice is rolled?

When there are two separate events, we can represent the possible outcomes using a grid:

coin toss \ dice roll	1	2	3	4	5	6
H	×	×	×	⊗	×	×
T				○		

Crosses have been used to highlight the favourable coin outcomes (H) and circles for the favourable dice outcomes (4).

The combined favourable outcome (H and 4) occurs only once, out of 12 possible outcomes, so:

$$P(\text{H or } 4) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{1}{12}$$

Now, the individual probabilities, $P(H) = \frac{1}{2}$ and $P(4) = \frac{1}{6}$

Notice that $P(\text{H or } 4) = \frac{1}{12} = \frac{1}{2} * \frac{1}{6} = P(H) * P(4)$

Now, the result of tossing a coin does not influence the result of rolling a dice. These are said to be *independent events* :

- Independent events are those which have no influence on each other

We see in the above calculation, that to find the probability of a Head AND a 4 that we MULTIPLY the probability of a Head by the probability of a 4. This is a general rule for independent events:

- If the individual probabilities of several *independent events* are P_1 , P_2 , P_3 , ... etc., then the probability P all the events occurring is given by *the multiplication law of probability* :

$$\boxed{P = P_1 * P_2 * P_3 * \ldots\ldots}$$

Example

What is the probability of rolling a 3 with a dice and drawing a jack from a pack of cards?

These are independent events, and:

$$P(3) = \frac{1}{6}$$

$$P(\text{jack}) = \frac{1}{13}$$

$$P(4 \text{ and a jack}) = P(3) * P(\text{jack}) = \frac{1}{6} * \frac{1}{13} = \frac{1}{78}$$

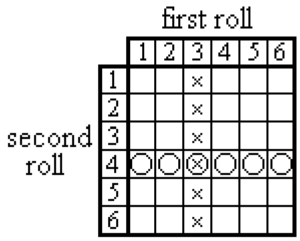
Example

What is the probability of rolling a 3 and then a 4 with a dice.

These are independent events and each has a probability of 1/6.

$$\text{The probability of both occuring, } P(3 \text{ and } 4) = \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

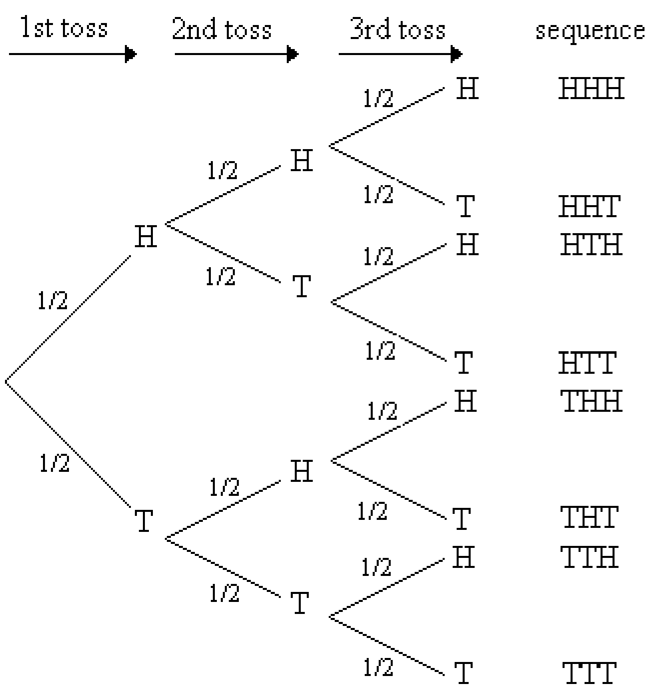
Again, we can work out the answer using the basic definition of probability. We need to work out all possible outcomes and identify the favourable ones:



There are 6*6 = 36 possible results and only one of them has 3 on the first roll and 4 on the second roll, so the probability is 1/36.

PROBABILITY TREE - start of this chapter - contents

The following represents tossing a coin three times. On the first toss, we may get a head (H) or a tail (T). After either of these we may then get a head or a tail, and after each of these another head or a tail. There are eight different possible sequences of heads and tails:



Notice that:

- we write the probability of each individual event on each branch

- the sum of probabilities on branches coming from the *same* point equals one (=½ + ½ in all the above cases) - this is because on each toss of a coin the possible outcomes are complementary, i.e. a head *or* a tail must occur

This above type of diagram is called a *probability tree* .

Each sequence, such as HHH, is a combined event, made up of 3 successive outcomes, and there are 8 such sequences in all.

Now, the result of a toss of a coin is *independent* of any previous results, so if we follow one sequence such as HHH, to work out its probability, we apply the *multiplication law* . So:

- P(HHH) = P(H) * P(H) * P(H) = ½ * ½ * ½ = 1/8
- P(HHT) = P(H) * P(H) * P(T) = ½ * ½ * ½ = 1/8 etc.

Also, if we flip a coin three times, only *one* of the above combined outcomes can occur. We could get HHH *or* HHT *or* HTH etc. So, the eight combined outcomes are *mutually exclusive* , and so to find the probability of one or other occurring we use the *addition law* :

For example, what is the probability of getting only one tail?

- There are three routes that contain only one tail: HHT HTH THH
- So, there are 3 favourable routes out of 8 possible routes, each with a probability of 1/8, so the probability of one of them occurring is 3/8

EXPERIMENTAL PROBABILITY - start of this chapter - contents

Sometimes probabilities are based on actual observations, and cannot be worked out theoretically, but we can still use the basic definition of probability.

Suppose that in testing light bulbs, on average 5 in 100 were found to be faulty. If one bulb is selected at random from a box, what is the probability of it being faulty?

$$P(\text{faulty}) = \frac{\text{favourable outcomes}}{\text{possible outcomes}} = \frac{5}{100} = 0.05 = 5\%$$

The answer is not exact in the way that P(head) = 0.5 was for a coin. We may actually find, for example, that there are 4 faulty bulbs in one batch of 100 bulbs and 6 in another batch.

Note that we still use the term 'favourable outcome' in the calculation, even though we may not think it 'favourable' to select a faulty bulb. The term 'favourable outcome' just refers to how many ways a particular outcome can occur.

Example

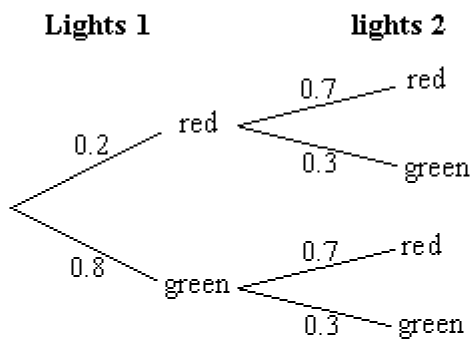
When Bob drives to work each day he passes two sets of traffic lights controlled by workmen. The lights are either red (for 'stop') or green (for 'go'). The following are the probabilities that the lights are on red:

- Lights 1, probability of red = 2/10
- Lights 2, probability of red = 7/10

(i) What is the probability that Bob will have to stop at both sets of lights?
(ii) What is the probability that he will not have to stop at either set?
(iii) What is the probability that he will only have to stop at one set?

The probability of having to stop at Lights 1 = 2/10 = 0.2
So, the probability of *not* having to stop at Lights 1 = 1 - 0.2 = 0.8

The probability of having to stop at Lights 2 = 7/10 = 0.7
So, the probability of *not* having to stop at Lights 2 = 1 - 0.7 = 0.3



(i) The probability of Bob having to stop at both sets of lights (the red-red sequence)

- = 0.2*0.7 = 0.14

(ii) The probability that he will not have to stop at either set of lights (the green-green sequence)

- = 0.8*0.3 = 0.24

(iii) The probability that he will only have to stop at one set (the red-green or green-red sequence)

- = (0.2*0.3) + (0.8*0.7) = 0.06 + 0.56 = 0.62

Notice that the *total probability* = 0.14 + 0.24 + 0.62 = 1, since one of them must occur.

CONDITIONAL PROBABILITY - start of this chapter - contents

This sort of probability arises when the probability of one outcome depends on a previous outcome. The multiplication law of probabilities can be applied to successive events, but the various individual probabilities need to be worked out first - using a probability tree can be helpful, as indicated next.

Example

Matthew leaves his jacket over a chair with sweets in the pockets:

- the left-hand pocket contains 7 blue and 3 red sweets
- the right-hand pocket contains 3 blue and 6 red sweets

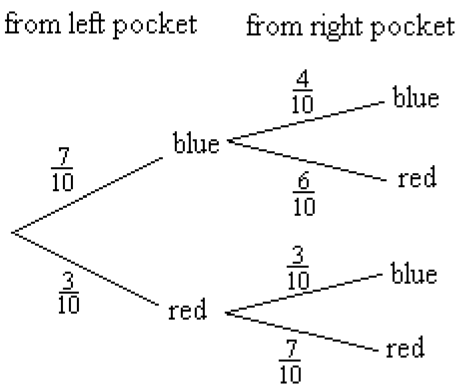
Matthew's brother Ryan takes one sweet at random from the left pocket and puts it in the right pocket.

If Matthew later takes a sweet from his right pocket, what is the probability of him getting a blue sweet?

We have to consider the effect of Ryan moving a red or a blue sweet from the left to the right pocket:

- if Ryan took a blue sweet from the left pocket and put in the right pocket, there are then 4 blue and 6 red sweets in the right pocket
- if Ryan took a red sweet from the left pocket and put in the right pocket, there are then 3 blue and 7 red sweets in the right pocket

We can represent all the possible events with a probability tree:



We are interested in the routes (i.e. sequences of events) that end in blue, which are the first and the third from the top.

As we follow any one of the four possible routes, we multiply their probabilities:

For the top route, P(blue, blue) = 7/10*4/10 = 28/100 =0.28
For the third route, P(red, blue) = 3/10*3/10 = 9/100 = 0.09

Only *one* of the four routes can occur, so the routes are mutually exclusive, and so we add their probabilities:

- the probability of Matthew taking a blue sweet from his right pocket = 0.28 + 0.09 = 0.37

Note: Work out the probabilities for the other two routes - if you add all four together you should get 1.
