Longman

Effective Guide to 'O' LEVEL

ADDITIONAL MATHEMATICS

- Based on the latest syllabus
- Comprehensive notes
- Worked examples
- Revision exercises
- Specimen examination papers
- Answers

PEARSON
Longman

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Contents

Introduction

In this second edition, new topics: Sets, Indices and Surds, Permutations and Combinations, and Matrices are included and given greater emphasis. For a more indepth coverage, Kinematics is now a chapter on its own.

In general, when you prepare for examinations, the first things you need to know will be the syllabus and the examinations structure. When you revise your work, make sure you cover all the content specified in the syllabus. For ease of reference, the content required in the syllabus for each topic is outlined at the beginning of each chapter.

It is advisable to cover a topic at one time and attempt questions related to the topic, questions from your textbooks, guide books as well as the Cambridge past year questions. It will also be better if you revise related topics consecutively.

In preparing for additional mathematics examinations, a crucial factor is **time.** It is insufficient to just revise and practise. You will need to sit for mock examinations too! After revising all the topics, it is advisable that you sit down will all the materials you need for an examination — question papers, formula list, calculator, A4 line papers, graph papers, stationery and a watch. Remember to put up a 'Do not disturb' sign too. Attempt an entire paper within the given time limit. You find that you will be better and faster after practising a few complete papers. By simulating the examination conditions, you be more confident and know exactly what to do during the actual examination.

The amount of time you should spend on each question is determined by the number of marks allocated to it. In the event that you are 'stuck' at a question, you may want to move on and solve other questions and attempt the unsolved question later. If you have time after answering all the questions, it will be advantageous to check your answers.

Remember to have an early night the day before the examinations so that you will be fresh and mentally alert. Well, good luck and happy revision.

Additional Mathematics Syllabus

Secondary

GCE 'O' Level

ADDITIONAL MATHEMATICS

G.C.E. O LEVEL ADDITIONAL MATHEMATICS

Syllabus Aims

The course should enable students

1. to extend their elementary mathematical skills and use these in the context of more advanced techniques;
2. to develop an ability to apply mathematics in other subjects, particularly science and technology;
3. to develop mathematical awareness; and the confidence to apply their mathematical skills in appropriate situations;
4. to extend their interest in mathematics and appreciate its power as a basis for specific applications.

Assessment Objectives

The examination will test the ability of candidates to

1. recall and use manipulative techniques;
2. interpret and use mathematical data, symbols and terminology;
3. comprehend numerical, algebraic and spatial concepts and relationships;
4. recognise the appropriate mathematical procedure for a given situation;
5. formulate problems into mathematical terms and select and apply appropriate techniques of solution.

Examination Structure

There will be two papers, each of 2 hours and each carries 80 marks. Content for Paper 1 and Paper 2 will not be dissected.

Each paper will consist of approximately 10–12 questions of various lengths. There will be no choice of question except that the last question in each paper will consist of two alternatives, only one of which must be answered. The mark allocation for the last question will be in the range of 10–12 marks.

Detailed Syllabus

Knowledge of the content of the Syndicate's Ordinary level Syllabus D, (or an equivalent Syllabus) is assumed. Ordinary level material which is not repeated in the syllabus below will not be tested directly but it may be required indirectly in response to questions on other topics.

THEME OR TOPIC	CURRICULUM OBJECTIVES
1. Set language and notation	Use set language and notation, and Venn diagrams to describe sets and represent relationships between sets as follows:
	$A = \{x: x \text{ is } A \text{ natural number}\}$ $B = \{(x, y): y = mx + c\}$ $C = \{x : A \le x \le b\}$ $D = \{a, b, c, ...\}$
	Understand and use the following notation:
	Union of A and B — $A \cup B$ Intersection of A and B — $A \cap B$ Number of elements in set A — $n(A)$ "... is an element of ..." — $\in$ "... is not an element of ..." — $\notin$ Complement of set A — A' The empty set — $\varnothing$ Universal set — ε A is a subset of B — $A \subseteq B$ A is a proper subset of B — $A \subset B$ A is a not a subset of B — $A \nsubseteq B$ A is a not a proper subset of B — $A \not\subset B$
2. Functions	Understand the terms function, domain, range (image set), one-one function, inverse function and composition of functions.
	Use the notation $f(x) = \sin x$, $f: x \to \lg x$, $(x > 0)$, $f^{-1}(x)$ and $f^2(x)$ $(= f(f(x)))$.
	Understand the relationship between $y = f(x)$ and $y = \lvert f(x) \rvert$, where $f(x)$ may be linear, quadratic or trigonometric.
	Explain in words why a given function is a function or why it does not have an inverse.
	Find the inverse of a one-one function and form composite functions.
	Use sketch graphs to show the relationship between A function and its inverse.

THEME OR TOPIC	CURRICULUM OBJECTIVES
3. Quadratic functions	Find the maximum or minimum value of the quadratic function $f : x \rightarrow ax^2 + bx + c$ by any method. Use the maximum or minimum value of $f(x)$ to sketch the graph or determine the range for a given domain. Know the conditions for $f(x) = 0$ to have (i) two real roots, (ii) two equal roots, (iii) no real roots; and the related conditions for a given line to (i) intersect a given curve, (ii) be a tangent to a given curve, (iii) not intersect a given curve. Solve quadratic equations for real roots and find the solution set for quadratic inequalities.
4. Indices and surds	Perform simple operations with indices and with surds, including rationalising the denominator.
5. Factors of polynomials	Know and use the remainder and factor theorems. Find factors of polynomials. Solve cubic equations.
6. Simultaneous equations	Solve simultaneous equations in two unknowns with at least one linear equation.
7. Logarithmic and exponential functions	Know simple properties and graphs of the logarithmic and exponential functions including $\ln x$ and e^x (series expansions are not required). Know and use the laws of logarithms (including change of base of logarithms). Solve equations of the form $a^x = b$.
8. Straight line graphs	Interpret the equation of a straight line graph in the form $y = mx + c$. Transform given relationships, including $y = ax^n$ and $y = Ab^x$, to straight line form and hence determine unknown constants by calculating the gradient or intercept of the transformed graph. Solve questions involving mid-point and length of line. Know and use the condition for two lines to be parallel and perpendicular.

THEME OR TOPIC	CURRICULUM OBJECTIVES
9. Circular measure	Solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure.
10. Trigonometry	Know the six trigonometric functions of angles of any magnitude (sine, consine, tangent, secant, cosecant, cotangent).
	Understand amplitude and periodicity and the relationship between graphs of e.g. sin x and sin $2x$.
	Draw and use the graphs of $y = a\sin(bx) + c$, $y = a\cos(bx) + c$, $y = a\tan(bx) + c$, where a, b are positive integers and c is an integer.
	Know and use the relationships $\frac{\sin A}{\cos A} = \tan A$, $\frac{\cos A}{\sin A} = \cot A$, $\sin^2 A + \cos^2 A = 1$, $\sec^2 A = 1 + \tan^2 A$, $\operatorname{cosec}^2 A = 1 + \cot^2 A$, and solve simple trigonometric equations involving the six trigonometric functions and the above relationships (not including general solution of trigonometric equations).
	Prove simple trigonometric identities.
11. Permutations and combinations	Recognise and distinguish between a permutation case and a combination case.
	Know and use the notation n!, (with 0! = 1), and the expressions for permutations and combinations of n items taken r at a time.
	Answer simple problems on arrangement and selection (cases with repetition of objects, or with objects arranged in a circle or involving both permutations and combinations, are excluded).
12. Binomial expansions	Use the Binomial Theorem for expansion of $(a + b)^n$ for positive integral n.
	Know and use the general term ${}^nC_r a^{n-r} b^r$, $0 < r \leq n$ (knowledge of the greatest term and properties of the coefficients is not required).

THEME OR TOPIC	CURRICULUM OBJECTIVES
13. Vectors in 2 dimensions	Use vectors in any form, e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$, $\overrightarrow{AB}$, $\mathbf{p}$, $a\mathbf{i} + b\mathbf{j}$. Know and use position vectors and unit vectors. Find the magnitude of a vector. Add and subtract vectors and multiply vectors by scalars. Compose and resolve velocities. Use relative velocity including solving problems on interception (but not closest approach).
14. Matrices	Display information in the form of matrix of any order and interpret the data in a given matrix. Solve problems involving the calculation of the sum and product (where appropriate) of two matrices and interpret the result. Calculate the product of a scalar quantity and a matrix. Use the algebra of 2 by 2 matrices (including the zero and identity matrix). Calculate the determinant and inverse of a non-singular matrix and solve simultaneous linear equations.
15. Differentiation and Integration	Understand the idea of a derived function. Use the notations f′(x), f″(x), $\frac{d^2y}{dx^2} = \left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$. Use the derivatives of the standard functions x^n (for any rational n), sin x, cos x, tan x, e^x, ln$x$, together with constant multiples, sums and composite functions of these. Differentiate products and quotients of functions. Apply differentiation to gradients, tangents and normals, stationary points, connected rates of change, small increments and approximations and practical maxima and minima problems. Discriminate between maxima and minima by any method.

THEME OR TOPIC	CURRICULUM OBJECTIVES
	Understand integration as the reverse process of differentiation. Integrate sums of terms in powers of x excluding $\frac{1}{x}$. Integrate functions of the form $(ax+b)^n$ (excluding $n=-1$), e^{ax+b}, $\sin(ax+b)$, $\cos(ax+b)$. Evaluate definite integrals and apply integration to the evaluation of plane areas. Apply differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of particles moving in a straight line (with variable or constant acceleration, and the use of $x-t$ and $v-t$ graphs).

MATHEMATICAL NOTATION

The list which follows summarizes the notation used in the Syndicate's Mathematics examinations. Although primarily directed towards Advanced level, the list also applies, where relevant, to examinations at other levels, i.e. O level, AO level.

Mathematical Notation

1. *Set Notation*

$\in$	is an element of
$\notin$	is not an element of
$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
$\{x: \ldots\}$	the set of all x such that ...
$n(A)$	the number of elements in set A
$\varnothing$	the empty set
$\mathcal{E}$	universal set
A'	the complement of the set A
$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
$\mathbb{Z}_n$	the set of integers modulo n, $\{0, 1, 2, \ldots, n-1\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \in \mathbb{Q} : x > 0\}$
$\mathbb{Q}_0^+$	the set of positive rational numbers and zero, $\{x \in \mathbb{Q} : x \geqslant 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \in \mathbb{R} : x > 0\}$
$\mathbb{R}_0^+$	the set of positive real numbers and zero, $\{x \in \mathbb{R} : x \geqslant 0\}$
$\mathbb{R}^n$	the real n tuples
$\mathbb{C}$	the set of complex numbers
$\subseteq$	is *a* subset of
$\subset$	is *a* proper subset of
$\nsubseteq$	is not *a* subset of
$\not\subset$	is not *a* proper subset of
$\cup$	union
$\cap$	intersection
(a, b)	the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$
(a, b)	the interval $\{x \in \mathbb{R} : a \leqslant x < b\}$
(a, b)	the interval $\{x \in \mathbb{R} : a < x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R} : a < x < b\}$

2. *Miscellaneous Symbols*

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to

$\approx$	is approximately equal to
$\cong$	is isomorphic to
$\propto$	is proportional to
$<$; $\ll$	is less than; is much less than
$\leqslant$; $\ngtr$	is less than or equal to or is not greater than
$>$; $\gg$	is greater than; is much greater than
$\geqslant$; $\nless$	is greater than or equal to or is not less than
∞	infinity

3. *Operations*

$a+b$	a plus b
$a-b$	a minus b
$a \times b$, ab, $a.b$	a multiplied by b
$a \div b$, $\frac{a}{b}$, a/b	a divided by b
$a:b$	the ratio of a to b
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
$\sqrt{a}$	the positive square root of the real number a
$\lvert a \rvert$	the modulus of the real number a
$n!$	n factorial for $n \in \mathbb{N}$ $(0! = 1)$
$\binom{n}{r}$	the binomial coefficient $\frac{n!}{r!(n-r)!}$, for n, $r \in \mathbb{N}$, $0 \leqslant r \leqslant n$ $\frac{n(n-1)\cdots(n-r+1)}{r!}$, for $n \in \mathbb{Q}$, $r \in \mathbb{N}$

4. *Functions*

f	function f
$f(x)$	the value of the function f at x
$f: A \to B$	f is a function under which each element of set A has an image in set B
$f: x \mapsto y$	the function f maps the element x to the element y
f^{-1}	the inverse of the function f
$g \cdot f$, gf	the composite function of f and g which is defined by $(g \cdot f)(x)$ or $gf(x) = g(f(x))$
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
Δx, δx	an increment of x
$\frac{dy}{dx}$	the derivative of y with respect to x
$\frac{d^n y}{dx^n}$	the nth derivative of y with respect to x
$f'(x)$, $f''(x)$, …, $f^{(n)}(x)$	the first, second, … , nth derivatives of $f(x)$ with respect to x

$\int y\,dx$	indefinite integral of y with respect to x
$\int_a^b y\,dx$	the definite integral of y with respect to x for values of x between a and b
$\frac{\partial y}{\partial x}$	the partial derivative of y with respect to x
$\dot{x}, \ddot{x}, \ldots$	the first, second, ... derivatives of x with respect to time.

5. *Exponential and Logarithmic Functions*

e	base of natural logarithms
e^x, exp x	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	natural logarithm of x
$\lg x$	logarithm of x to base 10

6. *Circular Functions and Relations*

sin, cos, tan cosec, sec, cot	the circular functions
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, cosec^{-1}, $\sec^{-1}$, $\cot^{-1}$	the inverse circular relations

7. *Matrices*

$\mathbf{M}$	a matrix $\mathbf{M}$
$\mathbf{M}^{-1}$	the inverse of the square matrix $\mathbf{M}$
det $\mathbf{M}$, $\lvert\mathbf{M}\rvert$	the determinant of the square matrix $\mathbf{M}$

8. *Vectors*

$\mathbf{a}$	the vector $\mathbf{a}$
$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment AB
$\hat{\mathbf{a}}$	a unit vector in the direction of the vector $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the cartesian coordinate axes
$\lvert\mathbf{a}\rvert$	the magnitude of $\mathbf{a}$
$\lvert\overrightarrow{AB}\rvert$	the magnitude of $\overrightarrow{AB}$
$\mathbf{a} \cdot \mathbf{b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \times \mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$

9. *Probability and Statistics*

A, B, C, etc.	events
$A \cup B$	union of the events A and B
$A \cap B$	intersection of the events A and B

Example 2

Make copies of the Venn diagram shown and shade the sets represented by each of the followings:

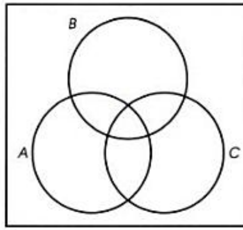
(a) $A \cap B$

(b) $A \cup C'$

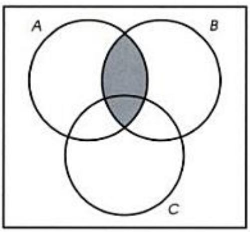
(c) $A \cup B \cup C$

(d) $(A \cup B) \cap C$

(e) $(A \cup B)' \cap C$

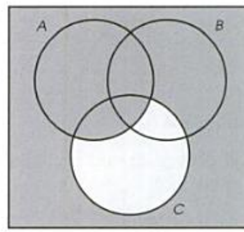


Solution (a)



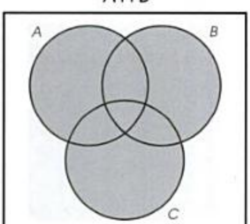
$A \cap B$

(b)



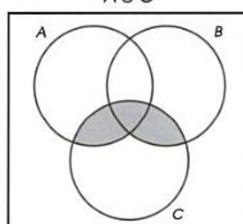
$A \cup C'$

(c)



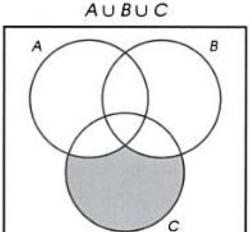
$A \cup B \cup C$

(d)



$(A \cup B) \cap C$

(e)



$(A \cup B)' \cap C$

Example 3

Given that $\varepsilon = \{1 \leq x \leq 20, x \in \mathbb{Z}\}$,

$A = \{x : x \text{ is a multiple of } 4\}$,
$B = \{x : x \text{ is a prime number}\}$,
$C = \{x : x \text{ is a multiple of } 3\}$

(a) List the elements of sets A, B and C.
(b) Find $n(A \cup B \cup C)$.
(c) List the elements of $A \cap B' \cap C$.

Solution (a) $A = \{4, 8, 12, 16, 20\}$
$B = \{1, 2, 3, 5, 7, 11, 13, 17, 19\}$
$C = \{3, 6, 9, 12, 15, 18\}$

(b) $(A \cup B \cup C) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19, 20\}$
$\therefore\ n(A \cup B \cup C) = 18$

(c) $(A \cap B' \cap C) = \{12\}$

Example 4

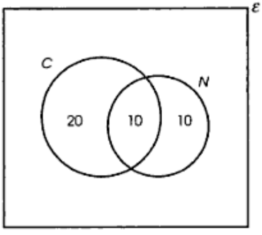
In a class of 40 students, 30 like to play computer games and 20 like to surf the net. It is given that

ε = {students in the class}
C = {students who like to play computer games}
N = {students who like to surf the net}

Let x be the number of students who neither like to play computer games nor surf the net.

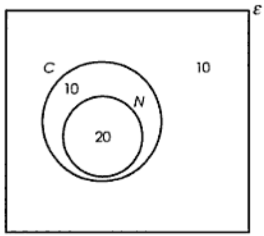
Using Venn diagrams, or otherwise, find
(a) the smallest possible value of x,
(b) the largest possible value of x.

Solution (a)



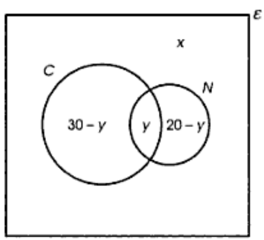
Smallest possible value of x is 0.

(b)



Largest possible value of x is $40 - 30 = 10$ when $N \subset C$.

Alternatively,



$x = n\,(C' \cap N')$

Let $y = n\,(C \cup N)$.

$$30 - y + y + 20 - y + x = 40$$
$$50 - y + x = 40$$
$$x = y - 10$$

(a) The smallest possible value of x is 0 when $y = 10$. It is impossible for x to be negative.

(b) The largest possible value of x occurs when y is maximum, i.e. when $y = 20$.
$\therefore$ largest possible value of $x = 20 - 10 = 10$.

Revision Exercises

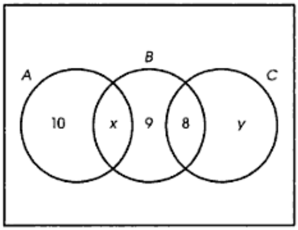
1. List the elements of each of the following sets:
   (a) $A = \{x : x \text{ is an odd number and } x < 10\}$,
   (b) $B = \{x : x \text{ is multiple of 5 and } x^2 < 300\}$,
   (c) $C = \{x : 1 < x < 5,\ x \in \mathbb{Z}^+\}$,
   (d) $D = \{x : 1 < 2x \leq 14,\ x \in \mathbb{Z}^+\}$.

2. A and B are two sets in the universal set. In separate Venn diagrams, shade the sets represented by each of the following:
   (a) $A \cup B$,
   (b) $A \cap B$,
   (c) $A \cup B'$,
   (d) $A \cap B'$,
   (e) $A' \cap B'$.

3. It is given that $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that
$A = \{x : x$ is an even number$\}$,
$B = \{x : x$ is a multiple of 3$\}$ and
$C = \{x : x$ is a multiple of 4$\}$.

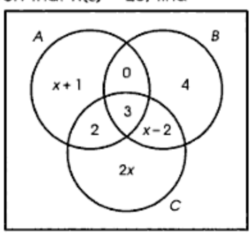
By drawing a Venn diagram, list the elements of the following sets:
(a) $A \cup B$,
(b) $A \cap B$,
(c) $A \cap B \cap C$,
(d) $(A \cup B \cup C)'$.

4. A, B and C are such that $\varepsilon = A \cup B \cup C$. The number of elements in each subset is shown in the Venn diagram.



(a) Given that $n(B \cap C) = n(B' \cap C)$, find the value of y.
(b) Find x and $n(\varepsilon)$, given that $n(A) = n(C)$.

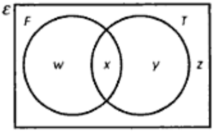
5. A, B and C are such that $\varepsilon = A \cup B \cup C$. The number of elements in each subset of is represented in the Venn diagram. Given that $n(\varepsilon) = 20$, find
(a) the value of x,
(b) $n(A \cap B')$,
(c) $n(B \cap C')$,
(d) $n(A' \cap B \cap C)$.



6. A hundred members of a health club were interviewed to find out whether they use free weights or treadmills more frequently. The outcome of the interview is represented by the Venn diagram below:

ε is the set of members interviewed,
F is the set of members who choose free weights,
T is the set of members who choose treadmills.

Given that $n(F) = 55$, $n(T) = 60$, find the maximum possible values of w, x and z.

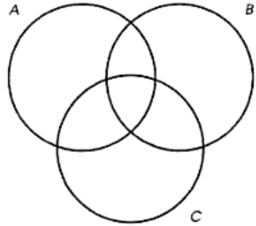


7. Members of a sea sports club may choose to participate in water-skiing, kayaking or wind-surfing.

31 members participate in water-skiing,
33 members participate in kayaking,
36 members participate in wind-surfing,
$2x$ members participate in water-skiing and kayaking,
13 members participate in kayaking and wind-surfing,
$x + 7$ members participate in water-skiing and wind-surfing,
x members participate in all three activities. It is compulsory that every member participate in at least one activity.

(a) Copy the Venn diagram and indicate the number of people in each subset.
(b) Given that there is a total of 70 club members,
   (i) find the value of x;
   (ii) find the number of members who participated in wind-surfing only;
   (iii) find the number of members who participated in water-skiing but not kayaking.

8. (a) On the Venn diagram below, shade the set $A \cup (B \cap C')$.



(b) There are 28 girls in a class. Of these, 17 sing in the choir and 15 play the piano. It is given that

ε = {girls in the class},

S = {girls who sing in the choir},

P = {girls who play the piano}.

(i) Find the smallest possible value of $n(S \cap P)$.

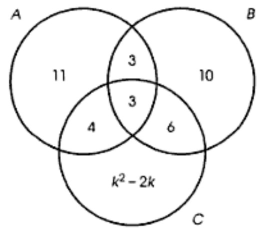
(ii) Express in set notation girls who neither sing in the choir nor play the piano. (C)

9. A, B and C are three sets and the numbers of elements are as shown in the Venn diagram. The universal set $\varepsilon = A \cup B \cup C$.

(a) State the value of $n((B \cup C)')$.

(b) If $x \in (A \cup B) \cap C$, find the probability that $x \in A$.

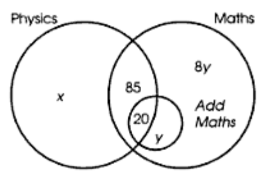
(c) If $n(C) = n(A)$, find the two possible values of k. (C)



10. In a school, some of the subjects that students can take are Mathematics, Additional Mathematics and Physics. The Venn diagram shows the combinations of these subjects that are possible, and the numbers and letters represent the number of students in each subset.

(a) Given that the number of students taking Physics is 123, calculate the value of x.

(b) Given that one sixth of those taking Mathematics also take Additional Mathematics, calculate the value of y and hence find the total number of students taking Mathematics. (C)



Chapter 2

Functions

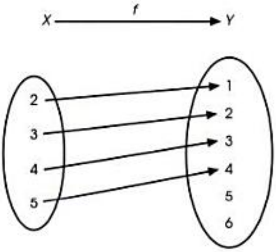
Curriculum Objectives:

- Understand the terms function, domain, range (image set), one-one function, inverse function and composition of functions
- Use the notation $f(x) = \sin x$, $f: x \to \lg x$, $(x > 0)$, $f^{-1}(x)$ and $f^2(x)$ $(= f(f(x)))$
- Understand the relationship between $y = f(x)$ and $y = |f(x)|$, where $f(x)$ may be linear, quadratic or trigonometric
- Explain in words why a given function is a function or why it does not have an inverse
- Find the inverse of a one-one function and form composite functions
- Use sketch graphs to show the relationship between a function and its inverse.

1. Functions

Consider set X and set Y, related by the function f, every element in the set X is mapped to a unique element in the set Y.

(2, 1) is an ordered pair where 1 is the image of 2.
The **domain** of the function is the set $X = \{2, 3, 4, 5\}$.
The **codomain** of the function is set $Y = \{1, 2, 3, 4, 5, 6\}$.



Notation

$f: X \to Y$ or $X \xrightarrow{f} Y$ denotes a function f from domain X to codomain Y.

$f: x \mapsto f(x)$ denotes a function f linking x, an element of the domain, to its image $y = f(x)$ in the codomain.

Considering each element in domain X,

$f(2) = 1$
$f(3) = 2$
$f(4) = 3$
$f(5) = 4.$ $\qquad\qquad\qquad \therefore f(x) = x - 1$

The **set of images** is called the **range** of f.

For example, the range of $f(x) = x - 1$ is $\{1, 2, 3, 4\}$. Elements like 5 and 6 in codomain Y are not images of any element in domain X and therefore are not in the range of f.

Example 1

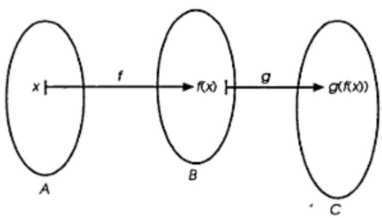
A function f is defined by $f : x \mapsto 3x$. Find the range of f for the domain $0 \le x \le 4$.

Solution $\quad 0 \le x \le 4$
$0 \le 3x \le 12$
$0 \le f(x) \le 12$

2. Composite Functions

When a function f is followed by another function g, we get a **composite function** gf.

$$gf(x) = g(f(x))$$



A is the domain and C is the codomain of the composite function gf.

Example 2

If $f : x \to x^2$ and $g : x \to x - 2$.

(i) Find the range of gf for the domain $-1 \le x \le 2$.

(ii) Find an expression for f^2.

Chapter 3

Quadratic Functions

Curriculum Objectives:

- Find the maximum or minimum value of the quadratic function $f: x \rightarrow ax^2 + bx + c$ by any method
- Use the maximum or minimum value of $f(x)$ to sketch the graph or determine the range for a given domain
- Know the conditions for $f(x) = 0$ to have
  (i) two real roots,
  (ii) two equal roots,
  (iii) no real roots,
  and the related conditions for a given line to
  (i) intersect a given curve,
  (ii) be a tangent to a given curve,
  (iii) not intersect a given curve
- Solve the quadratic equations for real roots and find the solution set for quadratic inequalities.

1. General form of a quadratic function

The general form of a quadratic function is $\mathbf{f(x) = ax^2 + bx + c}$

or $\quad y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Note : If $a = 0$, then we have a linear function which involves variables of power 1. Quadratic functions involve variables of highest power 2.

2. Maximum and minimum values of quadratic function $y = ax^2 + bx + c$

When $a > 0$, the function has a minimum value. The shape of its curve is $\cup$.
When $a < 0$, the function has a maximum value. The shape of its curve is $\cap$.

The maximum and minimum values can be found using two methods:

(i) "Completing the square"
Express the function in a form, like $\pm(ax + p)^2 + q$, (or other similar forms) where a, p and q are constants. The maximum or minimum is given by q and the corresponding value of x is $-\dfrac{p}{a}$.
(Note: You may need to use other similar forms, depending on the given quadratic function.)

(ii) Differentiation

Differentiate the function $y = ax^2 + bx + c$ with respect to x, to get $\frac{dy}{dx}$. Equate $\frac{dy}{dx}$ to zero and solve to get the x coordinate of the maximum/minimum value. Substitute this value into the expression for the function to get the corresponding value of y.

$\frac{dy}{dx} = 2ax + b = 0$

$\therefore x = -\frac{b}{2a}$

Substitute $x = -\frac{b}{2a}$ into $y = ax^2 + bx + c$

Example 1

Given that the curve whose equation is $y = p - (x - q)^2$ crosses the x-axis at the points (1, 0) and (3, 0), find

(i) the value of p and q,

(ii) the maximum value of y. (C)

Solution (i) For the point (1, 0),

$y = 0 = p - (1 - q)^2$

$p - (1 - 2q + q^2) = 0$

$p - 1 + 2q - q^2 = 0$ .... (1)

For the point (3, 0),

$y = 0 = p - (3 - q)^2$

$p - (9 - 6q + q^2) = 0$

$p - 9 + 6q - q^2 = 0$ .... (2)

(2) − (1)

$4q - 8 = 0$

$\therefore q = 2$

Substitute $q = 2$ into (1),

$p - 1 + 2(2) - (2)^2 = 0$

$p - 1 + 4 - 4 = 0$

$\therefore p = 1$

$\therefore p = 1, q = 2$

(ii) Use the "completing the square" method (this is a better and faster method to use in this question, especially so as the function is already expressed in the " squared" form)

Substituting the values of p and q into the function,

$y = 1 - (x - 2)^2$

By inspection, maximum value of $y = 1$.

Alternatively, using the differentiation method (not recommended but is shown to illustrate the method)

$y = 1 - (x - 2)^2 = 1 - x^2 + 4x - 4 = -x^2 + 4x - 3$

$\frac{dy}{dx} = -2x + 4 = 0$

$\therefore\ x = 2$

Substitute $x = 2$ into function,

$y = 1 - (2 - 2)^2$

$y = 1$

The maximum value of y is 1

Example 2

Express $y = \frac{1}{2}\{(x + 5)^2 + (x - 7)^2\}$ in the form $y = (x + q)^2 + r$. Hence find the least value of y and the corresponding value of x. (C)

Solution $y = \frac{1}{2}(x^2 + 10x + 25 + x^2 - 14x + 49)$

$= \frac{1}{2}(2x^2 - 4x + 74)$

$= x^2 - 2x + 37$

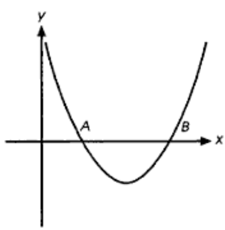
$= (x - 1)^2 + 36$

Maximum value of $y = 36$ when $x = 1$.

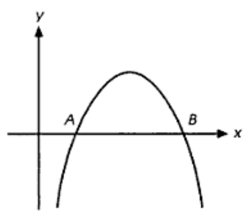
3. Graphs of quadratic functions

Shape of quadratic function $y = ax^2 + bx + c$ depends on a and $\boldsymbol{D}$, its **discriminant**, given by $\boldsymbol{b^2 - 4ac}$.

(i) When $a > 0$, $D > 0$



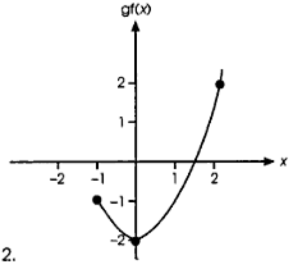
(ii) When $a < 0$, $D > 0$



The curves cut the x-axes at 2 different points A and B, i.e., the x-axes intersect the curves.

Solution (i) $gf(x) = g(f(x)) = g(x^2) = x^2 - 2$

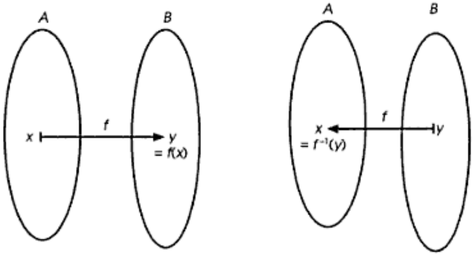
$gf(-1) = 1 - 2 = -1$
$gf(0) = 0 - 2 = -2$
$gf(2) = 4 - 2 = 2$



The range of gf is $-2 \leq gf(x) \leq 2$.

(ii) $f^2(x) = f(f(x)) = f(x^2) = x^4$
$\therefore f^2 : x \to x^4$

3. Inverse Function



To find the **inverse** of a function,
let $f(x) = y$, then $x = f^{-1}(y)$.

Example 3

The function f is defined by $f : x \mapsto 2x^2 - 1$. Find the expression for f^{-1}.

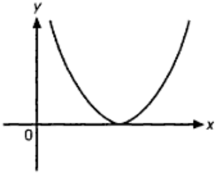
Solution Let $y = 2x^2 - 1$.

$$\sqrt{\frac{y+1}{2}} = x$$

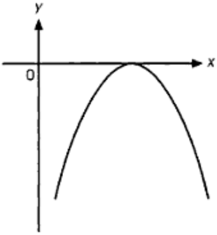
$$f^{-1}(y) = \sqrt{\frac{y+1}{2}}$$

$$f^{-1}(x) = \sqrt{\frac{x+1}{2}}$$

(iii) $a > 0$
$D = 0$

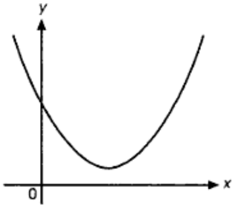


(iv) $a < 0$
$D = 0$

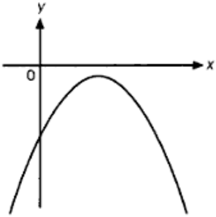


The curves touch the x-axes at one point, i.e. the x-axes are tangents to the curves.

(v) $a > 0$
$D < 0$



(vi) $a < 0$
$D < 0$



The curves are either entirely above or entirely below the x-axes, i.e. the x-axes do not intersect the curves.

To sketch a quadratic graph, consider the following:
(i) $a > 0$ or $a < 0$,
(ii) $D > 0$, $D = 0$ or $D < 0$,
(iii) the maximum or/and minimum points,
(iv) the y-intercept(s) (the value(s) of y when $x = 0$),
(v) the x-intercept(s) (the value(s) of x when $y = 0$).

4. General form of a quadratic equation

The general form of a quadratic equation is
$\boldsymbol{ax^2 + bx + c = 0}$ where a, b and c are constants and $a \neq 0$.

Solutions/roots to a quadratic equation are given by the following formula (which is given in the formula list).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

D, the discriminant of the equation = $b^2 - 4ac$, when

(i) $D > 0$, the equation has two real and unequal roots, A and B. (See figures i & ii.)

(ii) $D = 0$, the equation has two real and equal roots, i.e. one root A. (See figures iii & iv.)

(iii) $D < 0$, the equation has no real roots. (See figures v & vi.)

Example 3

Find the value of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has two equal roots. (C)

Solution Recall for an equation $ax^2 + bx + c = 0$ to have equal roots,

$D = b^2 - 4ac = 0$.

For the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ to have equal roots,

$D = (8p)^2 - 4(1 - 2p)(-(2 + 8p)) = 0$.

$64p^2 + (4 - 8p)(2 + 8p) = 0$

$64p^2 + 8 - 16p + 32p - 64p^2 = 0$

$16p + 8 = 0$

$p = -\frac{1}{2}$

5. Quadratic inequalities

When a quadratic function = 0, we get a quadratic equation $y = ax^2 + bx + c = 0$. Quadratic inequalities are obtained when $y = ax^2 + bx + c > 0$ or $y = ax^2 + bx + c < 0$. Partial inequalities are obtained when $y = ax^2 + bx + c \geq 0$ or $y = ax^2 + bx + c \leq 0$.

Solution of quadratic inequalities:

Method 1: Factorize the expression and consider the signs

(a) $(x + A)(x + B) > 0$

Both $(x + A)$ and $(x + B)$ have the same sign:

Either $(x + A) > 0$ and $(x + B) > 0$, or

$(x + A) < 0$ and $(x + B) < 0$

Draw two number lines and determine the range of x that satisfy both conditions $x + A > 0$ and $x + B > 0$ as well as that which satisfy both $(x + A) < 0$ and $(x + B) < 0$.

(b) $(x + A)(x + B) < 0$

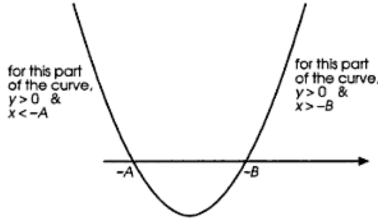
$(x + A)$ and $(x + B)$ have different signs:

Either $(x + A) > 0$ and $(x + B) < 0$, or $(x + A) < 0$ and $(x + B) > 0$

Draw two number lines and determine the range of x that satisfy both conditions $x + A > 0$ and $x + B < 0$ as well as that which satisfy both $(x + A) < 0$ and $(x + B) > 0$.

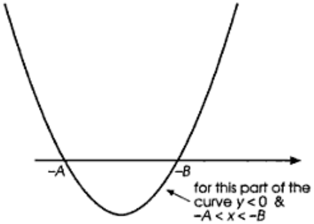
Method 2: Factorize the expression and sketch the curve

(a) $(x + A)(x + B) > 0$



Solution $x < -A$ or $x > -B$

(b) $(x + A)(x + B) < 0$



Solution $-A < x < -B$

Example 4

Determine the range of the values of x for which

(i) $x(x - 1) > 2$

(i) $x(x - 1) \leq 2$.

Solution (i) $x(x - 1) > 2$

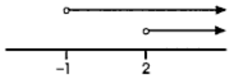
$x^2 - x - 2 > 0$

$(x - 2)(x + 1) > 0$

Method 1: Consider signs

For $(x - 2)(x + 1) > 0$, $(x - 2)$ and $(x + 1)$ have the same sign (arrows in number lines point in the same direction).

$(x - 2) > 0$ and $(x + 1) > 0$ or $(x - 2) < 0$ and $(x + 1) < 0$
$x > 2$ and $x > -1$ or $x < 2$ and $x < -1$



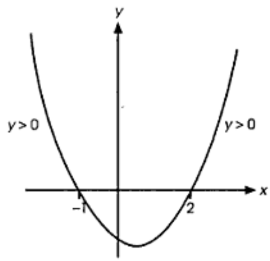
To satisfy both conditions $(x > 2$ & $x > -1)$, $x > 2$

To satisfy both conditions $(x < 2$ & $x < -1)$, $x < -1$

$\therefore$ The range of x for which $x(x - 1) > 2$ is $x > 2$ or $x < -1$.

Note: Use hollow dots ○→ for strict inequalities (i.e. > or <).
Use bold dots ●→ for partial inequalities that include the points (i.e. ≥ or ≤).

Method 2: Sketch curves
$(x - 2)(x + 1) > 0$



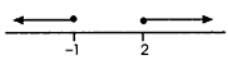
$x < -1$ or $x > 2$

(ii) $x^2 - x - 2 \leq 0$
$(x - 2)(x + 1) \leq 0$

Method 1: Consider signs

For $(x - 2)(x + 1) \leq 0$, $(x - 2)$ and $(x + 1)$ have different signs (i.e. arrows point in opposite directions).

$(x - 2) \geq 0$ and $(x + 1) \leq 0$ or $(x - 2) \leq 0$ and $(x + 1) \geq 0$
$x \geq 2$ and $x \leq -1$ or $x \leq 2$ and $x \geq -1$



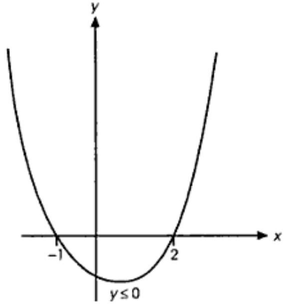
Both conditions ($x \leq 2$ and $x \leq -1$) cannot be satisfied.

To satisfy both conditions ($x \leq 2$ and $x \geq -1$), $-1 \leq x \leq 2$.

$\therefore$ The range for which $x(x - 1) \leq 0$ is $-1 \leq x \leq 2$.

Method 2: Sketch curves

$(x-2)(x+1) \leq 0$



$\therefore -1 \leq x \leq 2$

Example 5

Find the range of values of c for which the straight line $y = 2x + c$ intersects the curve $x^2 + y^2 = 20$ in two distinct points.

Solution At the points of intersection, the two equations must both be satisfied simultaneously.

$y = 2x + c$ .... (1)

$x^2 + y^2 = 20$ .... (2)

Substitute (1) into (2).

$x^2 + (2x + c)^2 = 20$

$x^2 + 4x^2 + 4xc + c^2 = 20$

$5x^2 + 4cx + c^2 - 20 = 0$

In order for the points of intersection of the line and the curve to be two distinct points, $5x^2 + 4xc + c^2 - 20 = 0$ has two real and distinct roots, i.e. its discriminant $D > 0$.

$D = b^2 - 4ac > 0$

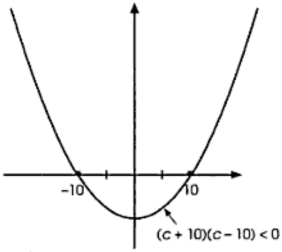
$(4c)^2 - 4(5)(c^2 - 20) > 0$

$16c^2 - 20c^2 + 400 > 0$

$4c^2 < 400$

$c^2 < 100$

$(c + 10)(c - 10) < 0$



From the graph, the range of values of c is $-10 < c < 10$.

Example 6

Find the value of k for which the line $y + 3x = k$ is a tangent to the curve $y = x^2 + 5$. (C)

Solution At the point of intersection, the two equations must be satisfied simultaneously.

$y + 3x = k \Rightarrow y = k - 3x$ .... (1)

$y = x^2 + 5$ .... (2)

Substitute (1) into (2).

$k - 3x = x^2 + 5$

$x^2 + 3x + 5 - k = 0$

For the line to be a tangent to the curve, i.e. for $x^2 + 3x + 5 - k$ to have one real root, its $D = 0$.

$D = b^2 - 4ac$

$D = (3)^2 - 4(1)(5 - k) = 0$

$9 - 20 + 4k = 0$

$k = \frac{11}{4}$

Example 7

Find the range of values of c for which $y = cx + 6$ does not meet the curve $2x^2 - xy = 3$. (C)

Solution At points of intersection, the two equations are simultaneously satisfied.

$y = cx + 6$ .... (1)

$2x^2 - xy = 3$ .... (2)

Substitute (1) into (2).

$2x^2 - x(cx + 6) = 3$

$2x^2 - cx^2 - 6x - 3 = 0$

When the line does not meet the curve, $2x^2 - cx^2 - 6x - 3 = 0$ has no real roots, i.e. $D < 0$.

$D = b^2 - 4ac = (-6)^2 - 4(2 - c)(-3) < 0$

$36 + 24 - 12c < 0$

$12c > 60$

$c > 5$

Revision Exercises

1. Find the maximum or minimum value of each of the following quadratic functions by expressing them in the form $\pm(ax + p)^2 + q$. State the corresponding value of x in each case.
   (a) $9x^2 + 24x + 14$
   (b) $40x - 16x^2 - 19$
   (c) $16 - 3x - x^2$
   (d) $3x^2 - 4x + 9$

2. Express $y = -2x^2 + 4x + 19$ in the form $y = (x + p)^2 + q$, where a, p and q are constants. Hence, state the maximum value of y and the corresponding value of x. Sketch the graph $y = -2x^2 + 4x + 19$.

3. Write $x^2 + kx + 64$ in the form $(x + p)^2 + q$ and obtain expressions for p and q in terms of k. Hence find the range of values of k such that $x^2 + kx + 64$ is positive for all values of x and deduce the corresponding range of values of k.

4. The following equations have equal roots. Find the value of p:
   (a) $x^2 + px + 4 = 0$
   (b) $x^2 - (p + 1)x = 5p - 30$
   (c) $(p - 1)x^2 + 4px + p + 20 = 0$.

5. The quadratic equation $px^2 + x + q = 0$ has roots 3 and -4. Find
   (i) the values of p and q,
   (ii) the range of values of r for which the equation $px^2 + x + q + r = 0$ has no real roots.

6. Find the range of values of k for which the equation $kx^2 + 2(2k + 2)x + 4k + 9 = 0$ has real roots.

7. Find the range of x which satisfy each of the following inequalities:
   (a) $(2x + 3)(x - 4) < 0$
   (b) $(7 - x)(x + 2) > 0$
   (c) $x^2 + 21 \geq 10x$
   (d) $2x^2 + 2x \leq x^2 - 5x + 8$
   (e) $\dfrac{3x^2 - 7x + 8}{2x^2 - 11x + 5} \geq 1$

8. Find the range of values of k for which the expression $2x^2 + 5x + k$ is positive for all real values of x.

9. Find the range of values of k for which the expression $2x(1 - x) + k$ is never positive for all real values of x.

10. Find the range of values of k for which the line $kx + y = 3$ intersects the curve $x^2 + 2y^2 = 8$ at two real and distinct points.

11. If the line $y = mx + c$ is a tangent to the curve $x^2 + y^2 = 10$, find the relationship between m and c.

12. Find the range of values of p for which the graph $y = px^2 + 9x + p + 12$ crosses the x-axis. State also the values of p for which the x-axis is a tangent to the curve.

13. (a) Calculate the range of values of x for which $x^2 + 4x - 5 > 5x - 3$
    (b) Calculate the range of values of c for which $3x^2 - 9x + c > 2.25$ for all values of x. (C)

14. (a) Find the range of values of x for which $x(10 - x) \geq 24$.
    (b) Find the value of k for which $2y + x = k$ is a tangent to the curve $y^2 + 4x = 20$. (C)

Indices and Surds

Curriculum Objectives:

- Perform simple operations with indices and with surds, including rationalising the denominator.

1. Indices

Multiplication of a by itself n number of times can expressed in the form of an index as a^n where a is the base and n is the index.

For example, $2^4 = 2 \times 2 \times 2 \times 2$
$= 16$

2. Laws of Indices

Laws of Indices	Notes
$a^m \times a^n = a^{m+n}$	
$a^m \div a^n = a^{m-n}$	
$(a^m)^n = a^{mn}$	
$a^m \times b^m = (a \times b)^m$	
$a^m \div b^m = \left(\frac{a}{b}\right)^m$	where $b \neq 0$
$a^0 = 1$	where $a \neq 0$ and is a real number
$a^{-m} = \frac{1}{a^m}$	where $a \neq 0$
$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$	where $a \neq 0$
$a^{\frac{1}{m}} = \sqrt[m]{a}$	
$a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}$	
$(a^m \times b^n)^l = a^{ml} \times b^{nl}$	

Example 1

Simplify (i) $3^2 \times 3^6$

(ii) $3^{x+y} \times 3^{2x-y}$

Solution (i) Method 1:

$3^2 \times 3^6 = (3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3 \times 3)$

$= 3^8$

Method 2:

Using $\boxed{a^m \times a^n = a^{m+n}}$

$3^2 \times 3^6 = 3^{2+6} = 3^8$

Note: Method 1 is used to verify the Law; it is faster and less cumbersome to use the Law of Indices.

(ii) $3^{x+y} \times 3^{2x-y} = 3^{(x+y)+(2x-y)} = 3^{3x}$

Example 2

Simplify (i) $3^5 \div 3^2$

(ii) $6^{x+y} \div 6^{x-y}$

Solution (i) $3^5 \div 3^2 = \dfrac{3^5}{3^2}$

$= \dfrac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3}$

$= 3 \times 3 \times 3$

$= 3^3$

Alternatively,

Using $\boxed{a^m \div a^n = a^{m-n}}$

$3^5 \div 3^2 = 3^{5-2} = 3^3$

(ii) $6^{x+y} \div 6^{x-y} = 6^{(x+y)-(x-y)} = 6^{2y}$

Example 3

Simplify (i) $(3^4)^3$

(ii) $(2^{12})^x$

Solution (i) $(3^4)^3 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3)$

$= 3^{12}$

Using $\boxed{(a^m)^n = a^{mn}}$

$(3^4)^3 = 3^{4 \times 3} = 3^{12}$

(ii) $(2^{12})^x = 2^{12x}$

Example 4

Simplify (i) $2^5 \times 3^5$

(ii) $3^2 \times x^2$

Solution (i) $2^5 \times 3^5 = (2 \times 2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3 \times 3)$

$= (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3)$

$= (2 \times 3)^5$

Alternatively,

Using $\boxed{a^m \times b^m = (a \times b)^m}$

$2^5 \times 3^5 = (2 \times 3)^5$

(ii) $3^2 \times x^2 = (3 \times x)^2$

Example 5

Simplify (i) $3^5 \div 2^5$

(ii) $x^6 \div 2^6$

Solution (i) $3^5 \div 2^5 = \dfrac{3 \times 3 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2 \times 2}$

$= \dfrac{3}{2} \times \dfrac{3}{2} \times \dfrac{3}{2} \times \dfrac{3}{2} \times \dfrac{3}{2}$

$= \left(\dfrac{3}{2}\right)^5$

Alternatively

Using $\boxed{a^m \div b^m = \left(\dfrac{a}{b}\right)^m}$

$3^5 \div 2^5 = \left(\dfrac{3}{2}\right)^5$

(ii) $x^6 \div 2^6 = \left(\dfrac{x}{2}\right)^6$

Consider a special case of the law $a^m \div a^n = a^{m-n}$ when $m = n$, for example,

$$a^3 \div a^3 = \frac{a \times a \times a}{a \times a \times a} = 1$$

and $a^3 \div a^3 = a^{3-3} = a^0$

Hence, we get the zero index law: $\boxed{a^0 = 1}$

Example 6

Simplify (i) 100^0

(ii) $\left(\frac{xy}{2}\right)^0$

Solution Using $\boxed{a^0 = 1}$

(i) $100^0 = 1$

(ii) $\left(\frac{xy}{2}\right)^0 = 1$

Consider another case of the law $a^m \div a^n = a^{m-n}$ where m < n, for example,

$$a^3 \div a^6 = \frac{a \times a \times a}{a \times a \times a \times a \times a \times a}$$

$$= \frac{1}{a \times a \times a}$$

$$= \frac{1}{a^3}$$

and $a^3 \div a^6 = a^{3-6} = a^{-3}$

Hence, we get the negative index law

$$\boxed{a^{-m} = \frac{1}{a^m}}$$

Example 7

Simplify (i) 3^{-2}

(ii) x^{-5}

Solution Using $\boxed{a^{-m} = \frac{1}{a^m}}$

(i) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(ii) $x^{-5} = \frac{1}{x^5}$

Next, consider $\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2}$

$$= \frac{1}{\frac{3 \times 3}{4 \times 4}}$$

$$= \frac{4 \times 4}{3 \times 3}$$

$$= \left(\frac{4}{3}\right)^2$$

Hence we obtain the following law of indices

$$\boxed{\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m}$$

Example 8

Simplify (i) $\left(\frac{5}{2}\right)^{-3}$

(ii) $\left(\frac{x}{3}\right)^{-y}$

Solution Using $\boxed{\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m}$

(i) $\left(\frac{5}{2}\right)^{-3} = \left(\frac{2}{5}\right)^3$

$$= \frac{8}{125}$$

(ii) $\left(\frac{x}{3}\right)^{-y} = \left(\frac{3}{x}\right)^y$

Consider an expression

$$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2} \times 2} = a$$

Take square roots on both sides of the equation:

$$a^{\frac{1}{2}} = \sqrt{a}$$

Likewise,

$$(a^{\frac{1}{m}})^m = a^{\frac{1}{m} \times m} = a$$

Take m^{th} root on both sides of the equation

$$\boxed{a^{\frac{1}{m}} = \sqrt[m]{a}}$$

Example 9

Find the value of (i) $125^{\frac{1}{3}}$

(ii) $\left(\frac{27}{8}\right)^{\frac{1}{3}}$

Solution Using $\boxed{a^{\frac{1}{m}} = \sqrt[m]{a}}$

(i) $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

(ii) $\left(\frac{27}{8}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}$

Example 10

Find the value of (i) $4^{\frac{3}{2}}$

(ii) $81^{\frac{3}{4}}$

Solution (i) $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = \left(\sqrt{4}\right)^3 = 2^3 = 8$

or $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = \sqrt{4^3} = \sqrt{64} = 8$

Hence the law $\boxed{a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}}$

(ii) $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3 = 3^3 = 27$

Example 11

Simplify (i) $(3^4 \times 2^3)^2$

(ii) $(4^2 \times 5^x)^3$

Solution (i) $(3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 2)$

$= 3^8 \times 2^6$

Alternatively,

Using $\boxed{(a^m \times b^n)^l = a^{ml} \times b^{nl}}$

$(3^4 \times 2^3)^2 = 3^{4\times 2} \times 2^{3\times 2}$

$= 3^8 \times 2^6$

(ii) $(4^2 \times 5^x)^3 = 4^{2\times 3} \times 5^{3x}$

$= 4^6 \times 5^{3x}$

Example 12

Simplify (i) $6^2 \times (2^2)^3 \div 3^2$

(ii) $64^{\frac{1}{6}} \times 3^2 \times 64^{\frac{1}{2}}$

(iii) $27^{\frac{2}{3}} \div 16^{\frac{1}{4}} \times 3^{-2}$

Solution (i) $6^2 \times (2^2)^3 \div 3^2$

$= 6^2 \div 3^2 \times (2^2)^3$ $\left(\text{using } a^m \div b^m = \left(\frac{a}{b}\right)^m\right)$

$= \left(\frac{6}{3}\right)^2 \times (2^2)^3$

$= 2^2 \times 2^6$ (using $(a^m)^n = a^{mn}$)

$= 2^8$ (using $a^m \times a^n = a^{m+n}$)

$= 256$

(ii) $64^{\frac{1}{6}} \times 3^2 \times 64^{\frac{1}{2}}$

$= 64^{\frac{1}{6}} \times 64^{\frac{1}{2}} \times 3^2$

$= 64^{\left(\frac{1}{6}+\frac{1}{2}\right)} \times 3^2$ (using $a^m \times a^m = a^{m+n}$)

$= 64^{\frac{2}{3}} \times 3^2$ (using $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$)

$= \left(\sqrt[3]{64}\right)^2 \times 3^2$

$= 4^2 \times 3^2$

$= (4 \times 3)^2$ (using $a^m \times b^m = (a \times b)^m$)

$= 12^2$

$= 144$

(iii) $27^{\frac{2}{3}} \div 16^{\frac{1}{4}} \times 3^{-2}$

$= 27^{\frac{2}{3}} \div \sqrt[4]{16} \times 3^{-2}$ $\left(\text{using } a^{\frac{1}{m}} = \sqrt[m]{a}\right)$

$= \left(\sqrt[3]{27}\right)^2 \div 2 \times 3^{-2}$ $\left(\text{using } a^{\frac{n}{m}} = (\sqrt[m]{a})^n\right)$

$= 3^2 \times 3^{-2} \div 2$ (using $a^m \times a^n = a^{m+n}$)

$= 3^{2-2} \div 2$

$= 3^0 \div 2$ (using $a^0 = 1$)

$= 1 \div 2$

$= \frac{1}{2}$

3. Surds

A **surd** is the square root of a number and it cannot be evaluated exactly.

A surd is an irrational number.

Recall:
An irrational number is one that cannot be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

Examples of surds are $\sqrt{2}, 2\sqrt{3}, 3+\sqrt{5}$ and $\frac{\sqrt{7}}{\sqrt{3}}$.

Fractions involving surds can be simplified by rationalising the denominator, i.e. to change the denominator which contains surd to a rational number.

Rationalising the denominator can be done by:
(i) multiplying both the numerator and denominator by the surd itself; or
(ii) multiplying both the numerator and denominator by its conjugate surd.

For example, the conjugate surd of $2\sqrt{5}+\sqrt{3}$ is $2\sqrt{5}-\sqrt{3}$.

Example 13

Simplify the following, by rationalising the denominators:

(a) $\frac{\sqrt{5}}{\sqrt{2}}$ (b) $\frac{4}{\sqrt{3}+2}$ (c) $\frac{3\sqrt{2}-\sqrt{3}}{2\sqrt{3}+\sqrt{2}}$

Solution (a)
$$\frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{10}}{2}$$

(b)
$$\frac{4}{\sqrt{3}+2} = \frac{4}{\sqrt{3}+2}\left(\frac{\sqrt{3}-2}{\sqrt{3}-2}\right) = \frac{4\sqrt{3}-8}{3-4} = 8-4\sqrt{3}$$

(c)
$$\frac{3\sqrt{2}-\sqrt{3}}{2\sqrt{3}+\sqrt{2}} = \frac{3\sqrt{2}-\sqrt{3}}{2\sqrt{3}+\sqrt{2}}\left(\frac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}-\sqrt{2}}\right) = \frac{6\sqrt{6}-2(3)-3(2)+\sqrt{6}}{4(3)-2} = \frac{7\sqrt{6}-12}{10}$$

Revision Exercises

1. Without using a calculator, simplify the following:

   (a) $\left(2^4 \times 4^0 \times 4^{-\frac{1}{2}}\right)^3$ (b) $5^2 \div 10^3 \times 2^2$

   (c) $(16^3 \times 4)^{\frac{1}{3}}$ (d) $\sqrt[3]{4} \times \sqrt[6]{32}$

   (e) $27^{\frac{1}{2}} \times 27^{\frac{1}{3}} \div 27^{\frac{1}{6}}$ (f) $\dfrac{5^{\frac{2}{3}} \times 25^{\frac{1}{3}}}{125^{\frac{4}{3}}}$

2. Solve the following equations:

   (a) $8^x = 2$ (b) $2^{x+2} = 4^{2x-2}$

   (c) $4^{3x} = \sqrt{32} \cdot 2^x$ (d) $\dfrac{3^{2x+1} \times 27^{x-2}}{81^{x-1}} = 1$

3. If $y = 3^x$, express $9^x + 3^{2x-1} - 81^x$ in terms of y.

4. Given that $y = ax^3 - b$, when $x = 2$, $y = 4$ and when $x = 3$, $y = 61$, find the value of a and b.

5. Show that $(3^5 \cdot x^{m-1})^3\left(\dfrac{1}{27x}\right)^m = (3^{15-3m})(x^{2m-3})$.

6. Simplify the following:

   (a) $\sqrt{27} + \sqrt{8}$ (b) $\sqrt{243} - 2\sqrt{27} + 3\sqrt{3}$

   (c) $\sqrt{32} \times \sqrt{6}$ (d) $\dfrac{\sqrt{42}}{\sqrt{6}}$

7. Simplify the following:

   (a) $(3 - \sqrt{7})(3 + \sqrt{7})$ (b) $(1 + \sqrt{3})^2$

   (c) $(3\sqrt{3} - 2)(4\sqrt{3} - 1)$ (d) $\dfrac{(5 - \sqrt{5})(3 + 2\sqrt{5})}{\sqrt{5}}$

8. Simplify the following by rationalising the denominators:

   (a) $\dfrac{4}{(2 - \sqrt{3})}$ (b) $\dfrac{3\sqrt{2} - \sqrt{5}}{3\sqrt{5} - \sqrt{2}}$

   (c) $\dfrac{4}{(\sqrt{2} + 3)^2}$ (d) $\dfrac{3 - \sqrt{5}}{\sqrt{3} + \sqrt{2}} + \dfrac{3 - \sqrt{5}}{\sqrt{3} - \sqrt{2}}$

Chapter 5

Factors of Polynomials

Curriculum Objectives:

- Know and use the remainder and factor theorems
- Find factors of polynomials
- Solve cubic equations

1. The Remainder Theorem

When a polynomial $f(x)$ (the dividend) is divided by $(ax - b)$ (the divisor), the remainder is $f(\frac{b}{a})$:
Dividend = Divisor × Quotient + Remainder

Example 1

Find the remainder when $f(x) = 4x^3 - 4x^2 - 7x - 2$ is divided by

(i) $x + 1$
(ii) $4x + 1$
(iii) $x - 2$.

Solution (i) $f(-1) = 4\,(-1)^3 - 4\,(-1)^2 - 7\,(-1) - 2 = -4 - 4 + 7 - 2 = -3$
(ii) $f(-\frac{1}{4}) = 4\,(-\frac{1}{4})^3 - 4\,(-\frac{1}{4})^2 - 7\,(-\frac{1}{4}) - 2 = -\frac{1}{16} - \frac{1}{4} + \frac{7}{4} - 2 = -\frac{9}{16}$
(iii) $f(2) = 4\,(2)^3 - 4\,(2)^2 - 7\,(2) - 2 = 32 - 16 - 14 - 2 = 0$

If the divisor is a factor of the dividend, then the remainder is zero.

2. The Factor Theorem

If a polynomial $f(x)$ (the dividend) is divided by $(ax - b)$ (the divisor) and $f(b/a) = 0$, then $ax - b$ is a factor of $f(x)$.

3. Factorization of Cubic Expressions

Cubic equations should have three linear factors, i.e., $f(x) = (x + a)\,(x + b)\,(x + c)$ and can be solved using a combination of methods of long division, inspection and trial and error using factor theorem.

Example 2

In example 1, $f(x) = 4x^3 - 4x^2 - 7x - 2$ has a factor $x - 2$. Given that $f(x)$ is exactly divisible by $(ax + 1)$. Find the value of a and hence completely factorize the expression.

Solution $4x^3 - 4x^2 - 7x - 2$ has a factor $x - 2$.

$$
\begin{array}{r}
4x^2 + 4x + 1 \\
x - 2 \overline{\smash{\big)}\, 4x^3 - 4x^2 - 7x - 2} \\
\underline{4x^3 - 8x^2 } \\
4x^2 - 7x \\
\underline{4x^2 - 8x } \\
x - 2 \\
x - 2
\end{array}
$$

$4x^3 - 4x^2 - 7x - 2$ is also exactly divisible by $(ax + 1)$.

$f(-\frac{1}{a}) = 4(-\frac{1}{a})^2 + 4(-\frac{1}{a}) + 1 = 0$

$\frac{4}{a^2} - \frac{4}{a} + 1 = 0$

$4 - 4a + a^2 = 0$

$(a - 2)^2 = 0$

$\therefore\ a = 2$

$\therefore$ another factor is $(2x + 1)$.

Factorizing $4x^2 + 4x + 1$ by inspection,

$4x^2 + 4x + 1 = (2x + 1)(2x + 1)$.

$\therefore\ 4x^3 - 4x^2 - 7x - 2 = (x - 2)(2x + 1)(2x + 1)$

Example 3

The remainder when $x^3 - 5x + a$ is divided by $x + 3$ is twice the remainder when it is divided by $x - 2$. Find the value of a. (C)

Solution Let $f(x)$ be $x^3 - 5x + a$ and R be the remainder when $x^3 - 5x + a$ is divided by $x - 2$.

$f(2) = (2)^3 - 5(2) + a = R$

$8 - 10 + a = R$

$a = R + 2$..........................(1)

$f(-3) = (-3)^3 - 5(-3) + a = 2R$

$-27 + 15 + a = 2R$

$a = 2R + 12$.....................(2)

(2) – (1),

$2R - R + 12 - 2 = 0$

$R = -10$

Substitute $R = -10$ into (1)

$a = -10 + 2 = -8$

Revision Exercises

1. Find the remainder when $6x^3 + 2x^2 + 4x + 3$ is divided by
   (a) $x + 3$
   (b) $2x - 1$
   (c) $3x + 4$

2. Find the value of a for which
   (a) $2x^3 - x^2 + 3x + a$ has a remainder of 10 when divided by $x - 2$,
   (b) $ax^5 + 2x^4 - 3x^2 - x + 6$ has a remainder of 7 when divided by $x - 1$,
   (c) $x^4 + (a - 1)x^3 + x^2 - 6x + 7$ has a remainder of 27 when divided by $x - 3$.

3. Completely factorize the following expressions:
   (a) $2x^3 - 3x^2 - 3x + 2$
   (b) $3x^3 - 8x^2 + 3x + 2$
   (c) $5x^3 - 5x = 2 - 2x^2$

4. Given that $f(x) = ax^3 + 5x^2 - 17x + b$ is exactly divisible by $3x + 1$ and has a remainder of 28 when divided by $x - 2$,
   (a) find the value of a and of b,
   (b) factorize $f(x)$ completely.

5. Given that the expression $2x^3 + ax^2 - 3x + b$ is exactly divisible by $x^2 + x - 2$, find the value of a and of b, and find the third factor of the expression.

6. The expression $x^3 + (b - 2a)x^2 + (a - b)x - 6$ has a factor of $x^2 + 4x + 3$. Find the value of a and of b.

7. Given that for all values of x, $6x^3 + 13x^2 - 9 = (ax + b)(x + 2)(2x - 1) + c$, evaluate a, b and c.

8. (a) Given that $4x^4 - 9a^2x^2 + 2(a^2 - 7)x - 18$ is exactly divisible by $2x - 3a$, show that $a^3 - 7a - 6 = 0$ and hence find the possible values of a.
   (b) The expression $2x^3 + bx^2 - cx + d$ leaves the same remainder when divided by $x + 1$ or $x - 2$ or $2x - 1$. Evaluate b and c. Given also that the expression is exactly divisible by $x + 2$, evaluate d. (C)

Simultaneous Equations

Curriculum Objectives:

- Solve simultaneous equations in two unknowns with at least one linear equation.

Simultaneous equations are equations that need to be solved at the same time.

1. Two simultaneous linear equations in two unknowns

To solve the equations, we usually eliminate one variable to find a solution to the second variable. The latter is then substituted into one of the equations to solve the first unknown. Usually there is only one solution to each unknown.

Example 1

The line $bx + ay = 11$ intersects $2ax + by = 15$ at the point (3, 1). Find the values of a and b.

Solution $x = 3$ and $y = 1$ satisfy the two simultaneous equations:

$b(3) + a(1) = 11$ ....(1)

$2a(3) + b(1) = 15$ ....(2)

From (1) $a + 3b = 11$

$a = 11 - 3b$..........(3)

From (2) $6a + b = 15$ ....(4)

To eliminate a, substitute (3) into (4)

$6(11 - 3b) + b = 15$

$17b = 51$

$b = 3$

Substitute $b = 3$ into (3)

$a = 11 - 3(3)$

$a = 2$

$\therefore\ a = 2$ and $b = 3$

2. Simultaneous linear and non-linear equations in two unknowns

When we deal with non-linear equations, there will be more than one solution. Apart from elimination by substitution, we may need to use factorization.
Using substitution, we start with the linear equation and express one unknown in terms of the other. Then we substitute it into the non-linear equation to eliminate one unknown.

To factorize a quadratic equation $ax^2 + bx + c = 0$, one may need to use the following formula (given in the formula list)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2

Solve the simultaneous equations

$\frac{x}{2} + \frac{y}{3} = 4$ .... (1)

$2xy = 45$ .... (2) (C)

Solution From (2)

$$x = \frac{45}{2y} \text{ .... (3)}$$

Substitute (3) into (1),

$$\frac{1}{2}\left(\frac{45}{2y}\right) + \frac{y}{3} = 4 \text{ .... (4)}$$

(4) $\times$ $12y$

$135 + 4y^2 = 48y$

$4y^2 - 48y + 135 = 0$

$$y = \frac{-(-48) \pm \sqrt{(-48)^2 - 4(4)(135)}}{2(4)}$$

$$y = \frac{48 \pm 12}{8} = \frac{15}{2} \text{ or } \frac{9}{2}$$

Substitute $y = \frac{15}{2}$ into (3), $x = \frac{45}{2\left(\frac{15}{2}\right)} = 3$;

Substitute $y = \frac{9}{2}$ into (3), $x = \frac{45}{2\left(\frac{9}{2}\right)} = 5$

$\therefore$ the solutions are $\left(3, \frac{15}{2}\right)$ or $\left(5, \frac{9}{2}\right)$

Example 3

The line $y = 3x - 1$ intersects the curve $2x^2 + 2y^2 - x + y - 11 = 0$ at A and B. Calculate the coordinates of A and of B. (C)

Solution At A and B, both equations must be satisfied simultaneously.

$y = 3x - 1$ .... (1)

$2x^2 + 2y^2 - x + y - 11 = 0$ .... (2)

Substitute (1) into (2),

$2x^2 + 2(3x - 1)^2 - x + (3x - 1) - 11 = 0$

$2x^2 + 2(9x^2 - 6x + 1) - x + 3x - 1 - 11 = 0$

$2x^2 + 18x^2 - 12x + 2 - x + 3x - 12 = 0$

$20x^2 - 10x - 10 = 0$

$2x^2 - x - 1 = 0.$

$(2x + 1)(x - 1) = 0.$

$\therefore\ x = -\frac{1}{2}$ or $x = 1$

Substitute $x = -\frac{1}{2}$ into (1)

$\therefore\ y = 3(-\frac{1}{2}) - 1 = -\frac{5}{2}$

Substitute $x = 1$ into (1)

$\therefore\ y = 3(1) - 1 = 2$

The coordinates of A and of B are $(-\frac{1}{2}, -\frac{5}{2})$ and $(1, 2)$.

Revision Exercises

Solve the simultaneous equations:

1. $x^2 + y^2 = 13$
   $3x + y = 9$

2. $x(x + 2y) = -4$
   $2x + y = 2$

3. $2x + y = 1$
   $4x^2 - 4xy - 4y^2 = -20$

4. $2x^2 = 110 - 3y^2$
   $2x + y = 12$

5. $x(3 + y) = 30$
   $y(2 + 2x) = 28$

6. Given that $(a, 7)$ is a solution of the simultaneous equations
   $3x - y = 8$ and
   $bx^2 - xy + 9 = y^2$,
   find
   (i) the value of a and of b,
   (ii) the coordinates of the other solution.

7. The sum of the perimeters of two squares is 18 m and the sum of their areas is 16.25 m^2. Find the dimensions of the squares.

8. The line $2x + y = 2$ intersects the curve $4x^2 + y^2 = 20$ at A and B. Find the coordinates of A and of B.

9. Solve the simultaneous equations.
   $y = x^2 + 5x - 3$
   $2y = 3x - 2$ (C)

10. Solve the simultaneous equations.
    $x + y = xy$
    $2y = x + 2$ (C)

Chapter 7

Logarithmic and Exponential Functions

Curriculum Objectives:

- Know simple properties and graphs of the logarithmic and exponential functions including $\ln x$ and e^x (series expansions are not required)
- Know and use the laws of logarithms (including change of base of logarithms)
- Solve equations of the form $a^x = b$.

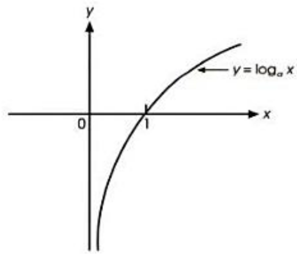
1. Logarithmic Function

General form is $\log_a x$ where $a > 0$ and x is a variable.

Important logarithmic functions:

(i) common logarithm where $a = 10$, $\log_{10} x$ i.e. $\lg x$.

(ii) natural logarithm where $a = e$, $\log_e x$ i.e. $\ln x$.



Properties of the graphs where $y = \lg x$ or $\ln x$:

(i) y does not exist for $x < 0$

(ii) $y = 0$ when $x = 1$

(iii) As $x \to \infty$ then $y \to \infty$

(iv) As $x \to 0$ then $y \to -\infty$

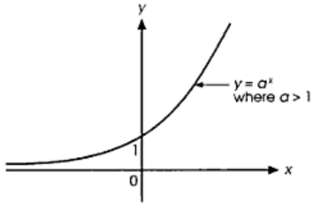
2. Exponential Functions

General form is a^x where a is a positive constant and x a variable.

Important exponential functions

(i) 10^x (inverse of $\lg x$)

(ii) e^x (inverse of $\ln x$)



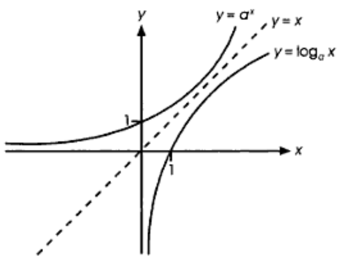
Properties of the graph $y = a^x$ ($a > 1$ and $x > 0$)

(i) $y > 0$ for all real values of x

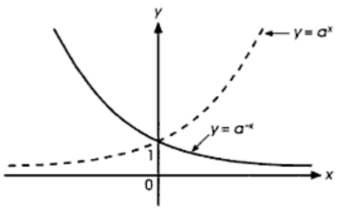
(ii) $y = 1$ when $x = 0$

(iii) As $x \rightarrow \infty$ then $y \rightarrow \infty$

(iv) As $x \rightarrow -\infty$ then $y \rightarrow 0$



Graph of $y = a^x$ is a reflection of $\log_a x$ graph in the line $y = x$.



$y = a^{-x}$ graph is obtained when $y = a^x$ is reflected in the y-axis.

3. Laws of Logarithm and Indices

Laws of Logarithm	Laws of Indices
$\log_a(xy) = \log_a x + \log_a y$	$a^m \times a^n = a^{m+n}$
$\log_a\left(\dfrac{x}{y}\right) = \log_a x + \log_a y$	$\dfrac{a^m}{a^n} = a^{m-n}$
$\log_a(x)^n = n\log_a x$	$(a^m)^n = a^{mn}$
$\log_a a = 1$	$a^0 = 1$
$\log_a 1 = 0$	$a^{-m} = \dfrac{1}{a^m}$
	$a^{\frac{1}{m}} = \sqrt[m]{a}$
	$a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}$

To change base, from base b to base a:

$$\log_a N = \frac{\log_b N}{\log_b a}$$

Example 1

(a) Solve the equation $2^x = 5$.

(b) Solve the equation $\lg x + \lg(3x + 1) = 1$.

(c) By using the substitution $y = e^x$, find the value of x such that $8e^{-x} - e^x = 2$.

(d) Given that $y = ax^b$, that $y = 2$ when $x = 3$ and that $y = \dfrac{2}{9}$ when $x = 9$, find the value of a and of b. (C)

Solution (a)

$2^x = 5$

$\lg 2^x = \lg 5$

$x \lg 2 = \lg 5$ since $\log_a(x)^n = n\log_a x$

$x = \dfrac{\lg 5}{\lg 2} = 2.32$

(b)

$\lg x + \lg(3x + 1) = 1$

$\lg x(3x + 1) = \lg 10$

$x(3x + 1) = 10$

$3x^2 + x - 10 = 0$

$(3x - 5)(x + 2) = 0$

$x = \dfrac{5}{3}$ or $x = -2$ (rejected as lg of negative constant is undefined)

$\therefore\ x = \dfrac{5}{3}$

(c) $8e^{-x} - e^{x} = 2$

Multiply throughout by e^x,

$8 - e^{2x} = 2e^x$

Using the substitution $y = e^x$,

$y^2 + 2y - 8 = 0$

$(y + 4)(y - 2) = 0$

$y = -4$ or $y = 2$

$e^x = -4$ (undefined) or $e^x = 2$

$\therefore\ x = 0.693$

(d) $y = ax^b$

When $x = 3$, $y = 2$;

$2 = a(3)^b$

$\lg 2 = \lg a + b \lg 3$ .... (1) (since $\log_a(xy) = \log_a x + \log_a y$ and $\log_a(x)^n = n \log_a x$)

When $x = 9$, $y = \frac{2}{9}$;

$\frac{2}{9} = a(9)^b$

$\lg \frac{2}{9} = \lg a + b \lg 9$

$\lg 2 - \lg 9 = \lg a + b \lg 9$ .... (2)

(1) – (2)

$\lg 9 = b(\lg 3 - \lg 9)$

$$\therefore b = \frac{\lg 9}{\lg 3 - \lg 9} = \frac{\lg 9}{\lg \frac{3}{9}} = -2$$

Substitute $b = -2$ into (1).

$\lg 2 = \lg a + (-2) \lg 3$

$\lg a = \lg 2 + 2\lg 3$

$\lg a = \lg 2 + \lg 3^2$

$\lg a = \lg 2 \times 3^2$

$\therefore\ a = 2 \times 3^2 = 18$

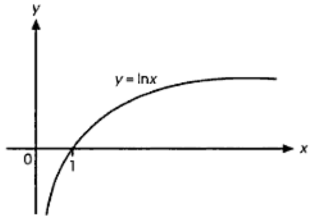
Example 2

(i) Sketch the graph of $y = \ln x$ for $x > 0$.

(ii) Express $x^2 = e^{x-2}$ in the form $\ln x = ax + b$.

(iii) Insert on your sketch the additional graph required to obtain a graphical solution of $x^2 = e^{x-2}$. (C)

Solution (i)



(ii) $x^2 = e^{x-2}$

$\ln x^2 = x - 2$

$2 \ln x = x - 2$

$\ln x = \frac{1}{2}x - 1$ where $a = \frac{1}{2}$ and $b = -1$

(iii) $x^2 = e^{x-2}$

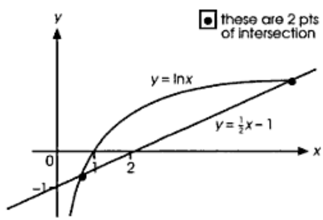
$\ln x = \frac{1}{2}x - 1$

Let $y = \ln x$ and $y = \frac{1}{2}x - 1$.

The intersection of $y = \ln x$ and $y = \frac{1}{2}x - 1$ graphs gives a solution of $x^2 = e^{x-2}$.

$y = \frac{1}{2}x - 1$ is a straight line graph with gradient $\frac{1}{2}$ and y-intercept of -1.

When $y = 0$, $x = 2$.



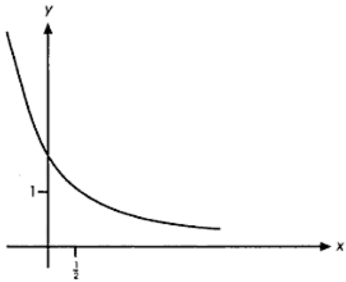
Example 3

Sketch the graph of $y = e^{1-2x}$.

Solution $y = e \times e^{-2x}$ Thus, it will take the shape of $y = e^{-2x}$.

$y = e^{1-2x} = 1$ when $1 - 2x = 0$ $\therefore x = \frac{1}{2}$

The graph of $y = e^{1-2x}$ is obtained by translating $y = e^{-2x}$ half an unit to the right.



Example 4

By changing bases, evaluate the following:

(a) $\log_2 5$

(b) $\log_{1.5} 2.6$

Solution

Note: By changing to base 10 or base *e*, it is possible to evaluate the above.

(a) $\log_2 5 = \dfrac{\lg 5}{\lg 2} = 2.32$

Alternatively,

$\log_2 5 = \dfrac{\ln 5}{\ln 2} = 2.32$

(b) $\log_{1.5} 26 = \dfrac{\lg 2.6}{\lg 2.5} = 2.36$

Alternatively,

$\log_{1.5} 2.6 = \dfrac{\ln 2.6}{\ln 2.5} = 2.36$

Revision Exercises

1. Solve the following equations:
   (a) $2 \lg x = \lg (x + 2)$
   (b) $2 \lg 2z + \lg (z + 1) - \lg 3z = 0$
   (c) $\lg (2x^2 + 5x + 2) = 3 \lg 2 + 1$

2. Solve the following equations:
   (a) $3^{y+1} = 7$
   (b) $2 \times 16^{x-1} = 8^x$
   (c) $2^x - 5 + 2^{2-x} = 0$

3. Solve the following equations:
   (a) $e^{3x-1} = 148$
   (b) $2e^{x-1} = 7$
   (c) $e^{2x} + 3e^{-2x} = 4$
   (d) $2e^{2x+1} - e^{x+1} - 6e = 0$

4. Given that $\lg 5 = p$, find in terms of p
   (i) 15
   (ii) $\lg 50$
   (iii) $\lg 25$

5. Given that $\ln 2 = 0.693$ and $\ln 9 = 2.197$, calculate without using calculator, the values of
   (i) $\ln 72$
   (ii) $\ln\sqrt{162}$
   (iii) $\ln 2.25$

6. Sketch the following functions:
   (i) $y = \lg 3x$
   (ii) $y = 3 \ln x$
   (iii) $y = \ln(x + 2)$

7. Sketch the following functions:
   (i) $y = 2e^{3x}$
   (ii) $y = e^{-x-1}$
   (iii) $y = \frac{1}{2}e^{\frac{1}{2}x-1}$

8. Using the substitution $y = 2^x$, solve the equation $4^x - 2^{x+2} + 3 = 0$.

9. (a) The curve with equation $py = qx$ passes through the points $(1, -12)$ and $(-2, \frac{3}{16})$. Find the value of p and of q.

(b) Solve the simultaneous equations:

$$\lg x + 2\lg y = 3$$
$$x^2 y = 125$$

(c) Given that $y = 2xe^{-x}$,

(i) show that y has a stationary value when $x = 1$.

(ii) Complete the following table.

x	0	0.5	1	2
y				

Using graph paper, draw the graph $y = 2x\,e^{-x}$ for $0 \le x \le 2$. By drawing a suitable line, find the solutions of the equation $x + 1 = 10xe^{-x}$. (C)

10. (a) The mass, m grams, of a radioactive substance, present at time t days after first being observed, is given by the formula

$$m = 24e^{-0.02t}.$$

Find

(i) the value of m when $t = 30$,

(ii) the value of t when the mass is half of its value at $t = 0$,

(iii) the rate at which the mass is decreasing when $t = 50$. (Answer this question after revising chapter 15: Differentiation)

(b) Solve the equation

$$\lg(20 + 5x) - \lg(10 - x) = 1.$$

(c) Given that $x = \lg a$ is a solution of the equation $10^{2x+1} - 7(10^x) = 26$, find the value of a. (C)

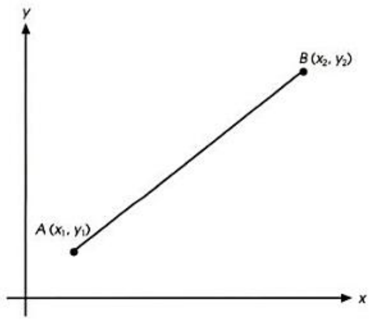
Chapter 8

Straight Line Graphs

Curriculum Objectives:

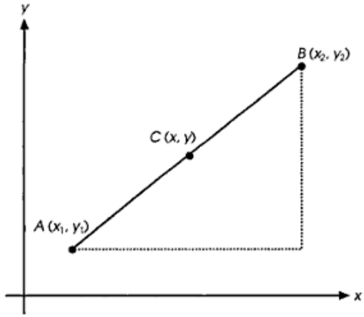
- Interpret the equation of a straight line graph in the form $y = mx + c$
- Transform given relationships, including $y = ax^n$ and $y = Ab^x$, to straight line form and hence determine unknown constants by calculating the gradient or intercept of the transformed graph
- Solve questions involving mid-point and length of a line
- Know and use the condition for two lines to be parallel and perpendicular.

1. Distance between two points



Given that the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively, distance between A and B, i.e. $\boldsymbol{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. Mid-point



Let the mid-point of AB be $C\ (x, y)$.

Coordinates of $C = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Example 1

Two points have coordinates $A\ (-8, 4)$ and $B\ (-2, 0)$, find (i) the distance between A and B; (ii) the coordinates of C, the mid-point of AB.

Solution (i) $AB = \sqrt{(-2-(-8))^2 + (0-4)^2}$

$= \sqrt{52}$

$= 7.21$ units

(ii) Mid-point $C = \left(\dfrac{-8+(-2)}{2}, \dfrac{4+0}{2}\right)$

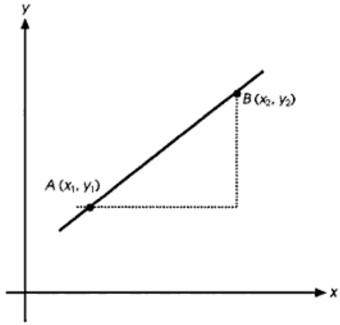
$= (-5, 2)$

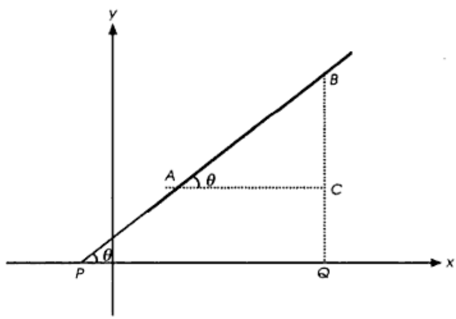
3. Gradient

Gradient, m, of a straight line joined by any two points is defined as

$$m = \frac{\text{the difference in the } y\text{-coordinates}}{\text{the difference in the } x\text{-coordinates}}$$

$$= \frac{\boldsymbol{y_1 \pm y_2}}{\boldsymbol{x_2 \pm x_1}} \text{ or } \frac{\boldsymbol{y_1 \pm y_2}}{\boldsymbol{x_1 \pm x_2}}$$

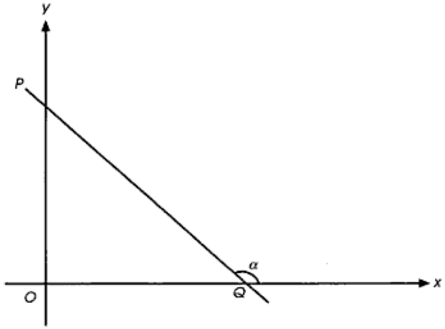




Gradient of AB, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ = tan $\angle BAC$ = tan θ

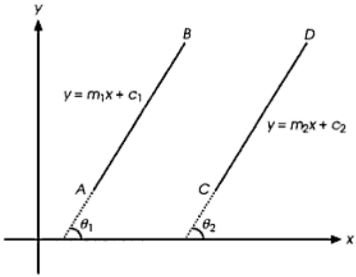
Since $\angle BAC = \angle BPQ$, gradient of a straight line = tangent of the angle which the line makes with the positive direction of the x-axis.

θ is acute $\therefore$ tan $\theta > 0$, gradient $\theta > 0$.



α is obtuse $\therefore$ $\tan \alpha < 0$, $\tan \alpha = -\tan \angle PQO$, gradient < 0.

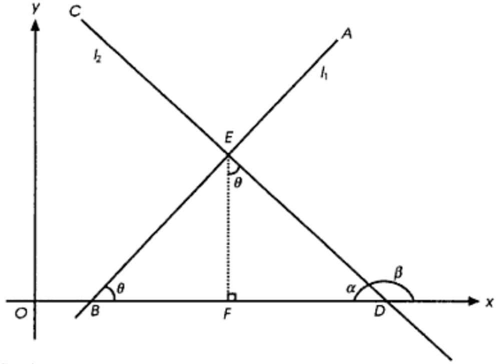
4. Gradients of two parallel lines



Equation of AB is $y = m_1x + c_1$
Gradient of $AB = m_1 = \tan \theta_1$
Equation of CD is $y = m_2x + c_2$
Gradient of $CD = m_2 = \tan \theta_2$
When AB is parallel to CD

$$\theta_1 = \theta_2$$
$$\tan \theta_1 = \tan \theta_2$$
$$\therefore \quad \boldsymbol{m_1 = m_2}$$

5. Gradients of two perpendicular lines



l_1 has gradient m_1

$m_1 = \tan \angle ABD = \tan \theta$

l_2 has gradient m_2

$m_2 = \tan \angle EDX = \tan \beta$

EF is the perpendicular from E to the x-axis.

$\angle DEF = \theta$

$\therefore\ m_1 = \dfrac{EF}{BF} = \tan\theta = \dfrac{DF}{EF}$

$m_2 = -\tan\alpha = -\dfrac{EF}{DF} = -\dfrac{1}{\tan\theta} = -\dfrac{1}{m_1}$

$\therefore\quad \boldsymbol{m_2 = \pm\dfrac{1}{m_1}}$

Example 2

Lines AC and BD are perpendicular to each other. Given that the gradient of AC is $\frac{1}{4}$, find the gradient of BD.

Solution Gradient of AC, m_{AC} is $\frac{1}{4}$.
Since AC and BD are perpendicular, gradient of BD is $-\frac{1}{m_{AC}} = -4$.

6. Area of Plane Figures

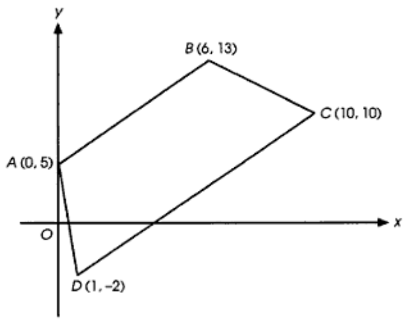
General formula for area of the figure with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$,.......$N(x_n, y_n)$, where $A, B, C...N$ are located in an anti-clockwise direction, is

$$\frac{1}{2}\begin{vmatrix} x_1\, x_2\, x_3 \ldots\ldots & x_n\, x_1 \\ y_1\, y_2\, y_3 \ldots\ldots & y_n\, y_1 \end{vmatrix}$$

$= \frac{1}{2}$((sum of all products $\searrow$)) − (sum of all products $\swarrow$))

Example 3

What is the area of the trapezium shown below?



Solution Area of trapezium $ADCB = \frac{1}{2}\begin{vmatrix} 0 & 1 & 10 & 6 & 0 \\ 5 & -2 & 10 & 13 & 5 \end{vmatrix}$

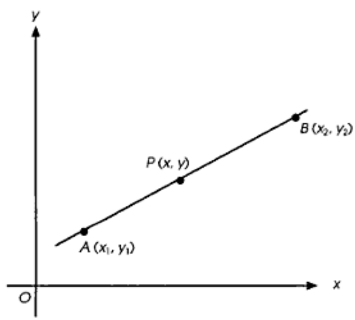
$= \frac{1}{2}\{[(0)(-2) + (1)(10) + (10)(13) + (6)(5)] - [(1)(5) + (10)(-2) + (6)(10) + (0)(13)]\}$

$= \frac{1}{2}[(0 + 10 + 130 + 30) - (5 - 20 + 60)]$

$= \frac{1}{2}(170 - 45) = 62.5 \text{ unit}^2$

7. Equation of a straight line

(i) When two points are given



To find the equation of the straight line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$, let $P(x, y)$ be a point lying on the same straight line.

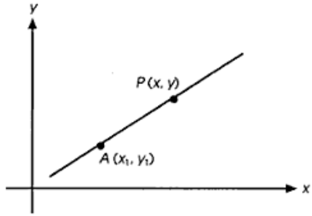
A, P and B are collinear,

$\therefore$ gradient of AP = gradient of AB

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\therefore \; \boldsymbol{y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)}$$

(ii) When one point and gradient of the straight line are given



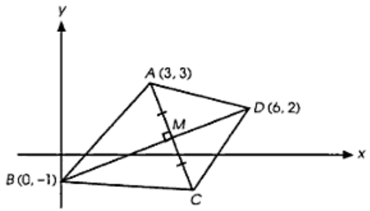
To find the equation of the straight line passing through a point $A(x_1, y_1)$ and with gradient m

Let $P(x, y)$ be a point lying on the straight line.

$$\frac{y-y_1}{x-x_1}=m$$

$$\therefore \; \boldsymbol{y-y_1=m(x-x_1)}$$

Example 4



In the quadrilateral $ABCD$, the points A, B and D are at (3, 3), (0, −1) and (6, 2) respectively. The line BD bisects the line CA at right angles at point M. Find the equation of BD and AC. Calculate

(i) the coordinates of M,

(ii) the coordinates of C,

(iii) the area of the quadrilateral $ABCD$ (C)

Solution Given two points $B(0, -1)$ and $D(6, 2)$, equation of BD is

$$\frac{y-(-1)}{x-0}=\frac{2-(-1)}{6-0}$$

$$\frac{y+1}{x}=\frac{3}{6}=\frac{1}{2}$$

$2y + 2 = x$

$2y = x - 2$

AC is perpendicular to BD.

From the equation of BD, the gradient of BD is $\frac{1}{2}$.

$\therefore$ gradient of $AC = -\frac{1}{m} = -2$.

Since gradient of AC is found and point $A(3, 3)$ is given, the equation of AC is

$$\frac{y-3}{x-3}=-2$$

$y - 3 = -2x + 6$

$y = 9 - 2x$

(i) Since M lies on both BD and AC, it must simultaneously satisfy the two corresponding equations of both lines.

$2y = x - 2$ .... (1)

$y = 9 - 2x$ .... (2)

From (1), $x = 2y + 2$ .... (3)

Substitute (3) into (2),

$y = 9 - 2(2y + 2) = 9 - 4y - 4$

$\therefore\ 5y = 5 \Rightarrow y = 1$

Substitute $y = 1$ into (3),

$x = 2 + 2 = 4$

$\therefore$ Coordinates of $M = (4, 1)$

(ii) Let the coordinates of C be (x_c, y_c)

M is the mid-point of AC, $\therefore\ M = \left(\frac{3+x_c}{2}, \frac{3+y_c}{2}\right) = (4, 1)$

Hence, $\frac{3+x_c}{2} = 4 \Rightarrow x_c = 8 - 3 = 5$

$\frac{3+y_c}{2} = 1 \Rightarrow y_c = 2 - 3 = -1$

$\therefore\ C = (5, -1)$

(iii) Area of $ABCD$ is $= \frac{1}{2}\begin{vmatrix} 3 & 0 & 5 & 6 & 3 \\ 3 & -1 & -1 & 2 & 3 \end{vmatrix}$

$= \frac{1}{2}\{((3)(-1) + (0)(-1) + (5)(2) + (6)(3))$

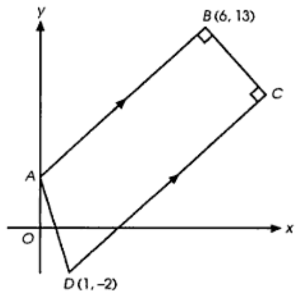
$\quad - ((0)(3) + (5)(-1) + (6)(-1) + (3)(2))\}$

$= \frac{1}{2}$((−3 Y;10 + 18) − (−5 − 6 + 6))

$= \frac{1}{2}(25 - (-5)) = 15$ units^2

Example 5

The diagram shows a trapezium $ABCD$ in which AB is parallel to DC. The point A lies on the y-axis. Points B and D are (6, 13) and (1, –2) respectively. Angles ABC and BCD are 90°. (C)



Given that the equation of DC is $3y = 4x - 10$, find

(i) the equation of AB,

(ii) the equation of BC,

(iii) the coordinates of A and of C,

(iv) the area of the trapezium.

Solution (i) AB is parallel to DC, from the equation of DC, it can be deduced that gradient of DC is $\frac{4}{3}$ $\therefore$ the gradient of AB is $\frac{4}{3}$.

$$\frac{y-13}{x-6}=\frac{4}{3}$$

$3(y-13) = 4(x-6)$

$3y - 39 = 4x - 24$

$3y = 4x + 15$

The equation of AB is $3y = 4x + 15$

(ii) BC is perpendicular to AB.

$\therefore$ gradient of BC is $-\frac{1}{m} = -\frac{1}{\frac{4}{3}} = -\frac{3}{4}$

$$\frac{y-13}{x-6}=-\frac{3}{4}$$

$4(y-13) = -3(x-6)$

$4y - 52 = -3x + 18$

$4y = 70 - 3x$

$\therefore$ equation of BC is $4y = 70 - 3x$

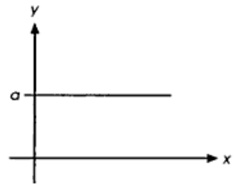
(iii) Given that A lies on the y-axis, let coordinates of A be $(0, y_A)$.
Using the equation of AB,
$3y = 4x + 15$
$3y_A = 4(0) + 15$
$\therefore\ y_A = 5$
$\therefore$ the coordinates of A is $(0, 5)$.

Point C is the point of intersection of both lines BC and DC, hence it must satisfy the two equations of BC and DC simultaneously. Substituting the coordinates of $C\,(x_c, y_c)$ into equations of lines BC and DC, we get:
$4y_c = 70 - 3x_c$ .... (1)
$3y_c = 4x_c - 10$ .... (2)
From (2), $y_c = \dfrac{4x_c - 10}{3}$ .... (3)
Substitute (3) into (1).
$$4\left(\frac{4x_c-10}{3}\right)=70-3x_c$$
$16x_c - 40 = 210 - 9x_c$
$25x_c = 250 \Rightarrow x_c = 10$
Substitute $x_c = 10$ into (3),
$\therefore\ y_c = \dfrac{4(10)-10}{3} = 10$
$\therefore$ the coordinates of $C = (10, 10)$

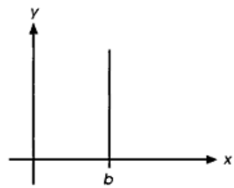
(iv) Area of the trapezium is $\dfrac{1}{2}\begin{vmatrix} 0 & 1 & 10 & 6 & 0 \\ 5 & -2 & 10 & 13 & 5 \end{vmatrix}$
$= \frac{1}{2}\,\{(0)(-2) + (1)(10) + (10)(13) + (6)(5) - ((1)(5) + (10)(-2) + (6)(10) + (0)(13))\}$
$= \frac{1}{2}(10 + 130 + 30 - (5 - 20 + 60))$
$= 62.5$ unit^2

Special equations parallel to the axes

(i) When a line is parallel to the x-axis, gradient, $m = 0$, y-intercept $= a$.
$y - a = 0\,(x - 0)$
$\therefore\ y = a$

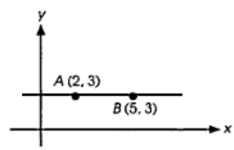


(ii) When a line is parallel to the y-axis, the x-intercept is b.
Gradient, m is undefined but $1/m = 0$
$1/m\,(y - 0) = x - b$
$\therefore\ x - b = 0 \Rightarrow x = b$



Example 6

Given that A and B are (2, 3) and (5, 3) respectively. Find the equation of AB.

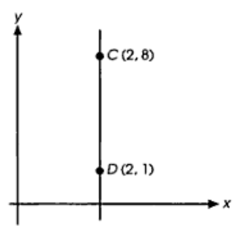


Solution Gradient of AB, $m_1 = \dfrac{3-3}{5-2} = 0$

Equation of AB is $y - 3 = 0\,(x - 2)$
$\therefore\ y = 3$

Example 7

Given that C and D are (2, 8) and (2, 1) respectively. Find the equation of CD.



Solution Gradient of CD, $m_2 = \dfrac{8-1}{2-2}$ = undefined

$$\frac{1}{m_2} = 0$$

Equation of CD is $y - 1 = m_2\,(x - 2)$
$\therefore\ x = 2$

8. Determination of unknowns from straight line graphs

A non-linear equation involving variables x and y can be converted to a straight line function in the form $Y = mX + c$ where X and Y are expressions in x and/or y, m is the gradient of X-Y graph and c the Y-intercept.

Non-linear equation	Straight line function	Y	X	M	c
$y = ax + bx^2$	$\frac{y}{x} = a + bx$	$\frac{y}{x}$	x	b	a
$y = \frac{a}{x} + b$	$y = \frac{a}{x} + b$	y	$\frac{1}{x}$	a	b
$y = \frac{a}{x-b}$	$\frac{1}{y} = \frac{1}{a}x - \frac{b}{a}$	$\frac{1}{y}$	x	$\frac{1}{a}$	$-\frac{b}{a}$
$y = \frac{a}{x^n} + b$	$y = \frac{a}{x^n} + b$	y	$\frac{1}{x^n}$	a	b
$\frac{1}{y} = ax^n + b$	$\frac{1}{y} = ax^n + b$	$\frac{1}{y}$	x^n	a	b
$xy = \frac{a}{x} + bx$	$y = \frac{a}{x^2} + b$	y	$\frac{1}{x^2}$	a	b
$x = bxy + ay$	$\frac{x}{y} = bx + a$	$\frac{x}{y}$	x	b	a
$y = ab^x$	$\lg y = \lg a + x\lg b$	$\lg y$	x	$\lg b$	$\lg a$
$y = ax^b$	$\lg y = \lg a + b\lg x$	$\lg y$	$\lg x$	b	$\lg a$
$y = \mathrm{n} - ax^b$	$\lg(n - y) = \lg a + b\lg x$	$\lg(n - y)$	$\lg x$	b	$\lg a$
$ya^x = b + n$	$\lg y = \lg(b + n) - x\lg a$	$\lg y$	x	$-\lg a$	$\lg(b + n)$

Table 8.1

Example 8

When the graph of y against $\sqrt{x}$ is drawn, a straight line is obtained which has gradient 2 and passes through the point (4, 7). Determine the relationship between x and y. Evaluate y when x is 25.

Solution $Y = mX + c$

$y = m\sqrt{x} + c$

Since gradient $m = 2$,

$y = 2\sqrt{x} + c$

When $x = 4$, $y = 7$.
$\therefore 7 = 2\sqrt{4} + c$
$\Rightarrow c = 7 - 2\sqrt{4} = 3$
$\therefore$ the relationship between x and y is $y = 2\sqrt{x} + 3$.

When $x = 25$,
$y = 2\sqrt{25} + 3 = 2(5) + 3 = 13$

Example 9

Variables x and y are related by the equation $\frac{x}{p} + \frac{y^2}{1} = 1$. When a graph of y^2 against x is drawn, the resulting line has a gradient of –2 and an intercept on the y^2 axis of 8. Calculate the value of p and of q. (C)

Solution $\frac{x}{p} + \frac{y^2}{q} = 1$

$\frac{y^2}{q} = -\frac{x}{p} + 1$

$y^2 = -\frac{q}{p}x + q$

The graph of y^2 against x is a straight line with gradient $-\frac{q}{p}$ and y intercept of q.

$\therefore$ gradient = $-\frac{q}{p} = -2 \Rightarrow \frac{q}{p} = 2$ .... (1)

and y Intercept = $q = 8$
Substitute $q = 8$ into (1).
$8/p = 2 \Rightarrow p = 4$
$\therefore$ $p = 4$ and $q = 8$

Example 10

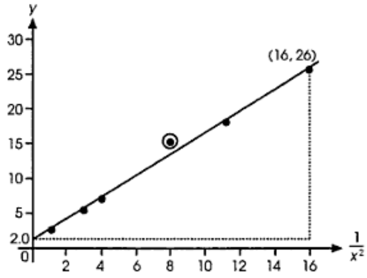
The table below shows experimental values of two variables x and y.

x	0.25	0.30	0.35	0.50	0.60	1.00
y	26.0	18.7	15.6	8.0	5.9	3.5

It is believed that one of the experimental values of y is abnormally large and also that the variables x and y are connected by an equation of the form $y = A + \frac{B}{x^2}$, where A and B are constants. By a suitable choice of variables this equation may be represented by a straight line graph. State these variables and, using the data given above, obtain corresponding pairs of values. Plot these values and hence identify the point corresponding to the abnormally large value of y. Ignoring this point use the remaining points to obtain a straight line graph. Use your line to evaluate A and B. (C)

Solution For equation $y = A + \frac{B}{x^2}$, a straight line graph is obtained by plotting y against $\frac{1}{x^2}$, with gradient B and y intercept A.

x	0.25	0.30	0.35	0.50	0.60	1.00
y	26.0	18.7	15.6	8.0	5.9	3.5
$\frac{1}{x^2}$	16.00	11.11	8.16	4.00	2.78	1.00



From the graph, the point with x and y coordinates (0.35, 15.6) has an abnormally large value of y.

Gradient of the line, $B = \frac{26 - 2.0}{16 - 0} = 1.5$

y intercept, $A = 2.0$

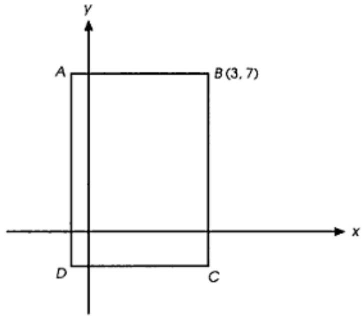
Revision Exercises:

1. The straight lines $y = ax - 6$, where a is a constant and $y = 2x + 3$ are perpendicular. State the value of a and hence find the coordinates of the point of intersection of the lines.

2. Two points A and B have coordinates (–3, –5) and (6, 2) respectively. Find the distance between A and B.
   Given that the perpendicular bisector of the line joining A and B meets the y-axis at C, calculate the coordinates of C.

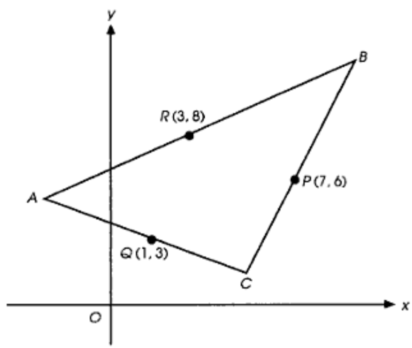
3. Find the equation of the line which passes through the point (1, –2) and is parallel to the line $y = 3x + 4$.

4. The diagram shows a rectangle $ABCD$, where equation of CD is $y = -2$. Given that B has coordinates (3, 7), find



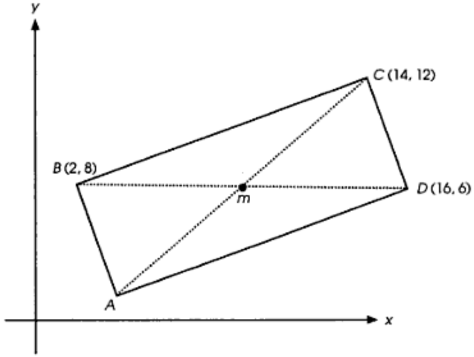
(i) the equation of AB,
(ii) the equation of BC,
(iii) the coordinates of C.
Given that CD is 4 units, find D and A and hence the area of $ABCD$.

5. In the diagram, $P(7, 6)$, $Q(1, 3)$ and $R(3, 8)$ are the mid-points of the sides of triangle ABC. Find



(i) the gradient of the line PQ,
(ii) the equation of the line AB,
(iii) the equation of the perpendicular bisector of the line AC.

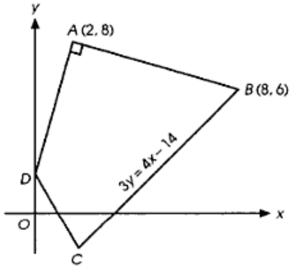
6. *ABCD* is a parallelogram whose diagonals meet at *M*. The coordinates of *B*, *C* and *D* are (2, 8), (14, 12) and (16, 6) respectively. Given that *AC* has a gradient 1, find the equation of *AC* and *BD*. Calculate



(i) the coordinates of *M*,
(ii) the coordinates of *A*.
Prove that *ABCD* is a rectangle. Find the area of the rectangle *ABCD*.

7. The straight lines $y = ax + 9$, where a is a constant, and $y = \frac{1}{2}x - 1$ are perpendicular. State the value of a and hence find the coordinates of the point of intersection of the lines. (C)

8.



The diagram shows a quadrilateral *ABCD* in which *A* is (2, 8) and *B* is (8, 6). The point *C* lies on the perpendicular bisector of *AB* and the point *D* lies on the *y*-axis. The equation of *BC* is $3y = 4x - 14$ and the angle $DAB = 90°$. Find
(i) the equation of *AD*,
(ii) the coordinates of *D*,

(iii) the equation of the perpendicular bisector of AB,

(iv) the coordinates of C.

Show that the area of the triangle ADC is 10 unit^2 and find the area of the quadrilateral $ABCD$. (C)

9. Express each of the following equations in the form $Y = mX + c$, where X and Y are expressions in X and/or y and m and c are constants. Hence, state the expressions for X, Y, m and c in each case.

(i) $y = 3x^2 + 2x$ (ii) $y = 6 - \frac{2}{x}$

(iii) $y = \frac{3}{\sqrt{x}} - \sqrt{x}$ (iv) $xy = 5x + \frac{2}{x}$

(v) $x^2 + 3y^2 = x$ (vi) $y = \frac{5}{x^4}$

(vii) $y = 4x^{\frac{4}{5}}$

10. Variables x and y are related by an equation $y = \frac{a}{b^x}$ where a and b are constants. When lg y is plotted against x, a straight line is obtained with a gradient of –2 and an intercept on the lg y-axis of 1. Calculate the value of a and of b.

11. The table shows experimental values of two variables x and y which are known to be connected by the equation of the form $ax^3 + by = x^2$. Explain how a straight line graph may be drawn to represent the given equation. Use the above data to plot the graph and hence estimate the value of a and of b.

x	1	2	3	4	5
y	3	20	63	144	275

12. It is known that x and y are related by the equation $x + a = \frac{b}{y}$, where a and b are constants.

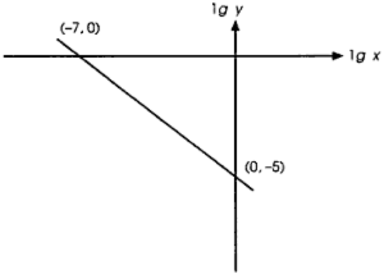
(i) Plot xy against y to obtain a straight-line graph.

(ii) Use your graph to estimate the value of a and b.

(iii) When graph of $\frac{1}{y}$ vs x is plotted, what is the value of the gradient of the straight line obtained and calculate the intercept in the $\frac{1}{y}$-axis.

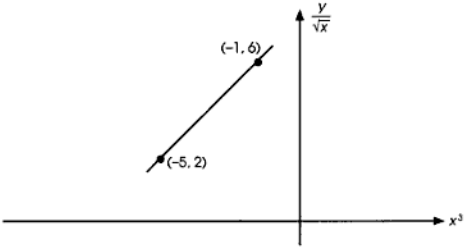
x	3	4	5	6	7
y	–3	–1.5	–1	–0.75	–0.6

13. The diagram shows the straight line graph obtained by plotting lg y against lg x. Given that the variables x and y are connected by the equation $y = ax^b$ where a and b are constants.



(i) Express lg y in terms of lg x.
(ii) Express y in terms of x.
(iii) Find the values of a and b.
(iv) Calculate the value of y when lg $x = 3$.

14. Variables x and y are related in such a way that when $\frac{y}{\sqrt{x}}$ is plotted against x^3, a straight line is obtained . This line passes through the points (–1 , 6) and (–5, 2) as shown on the diagram. Find



(i) an expression for y in terms of x,
(ii) the value of y when $x = 1$.

15. (a) The table shows experimental values of two variables, x and y.

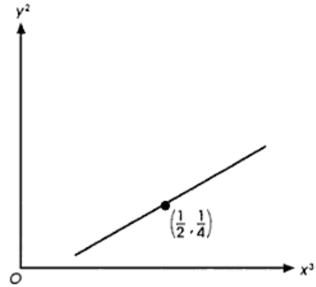
x	0.5	2.2	4.0	5.9	7.8
y	–7.5	–4.4	3.1	22.4	68.2

It is known that x and y are related by the equation $y + 10 = Ak^x$, where A and k are constants. Using graph paper, plot lg $(y + 10)$ against x for the above data and use your graph to estimate

(i) the value of A and of k,

(ii) the value of x when $y = 0$.

(b) Variables x and y are related by the equation $px^3 + qy^2 = 1$. The diagram shows the straight-line graph of y^2 against x^3 which passes through the point $\left(\frac{1}{2}, \frac{1}{4}\right)$.



(i) Given that the gradient of this line is $\frac{3}{4}$, calculate the value of p and of q.

(ii) Given also that this line passes through $(k, 4)$, find the value of k.

(C)

Chapter 9

Circular Measure

Curriculum Objective:

- Solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure.

1. Radian

Angles can be measured in terms of (i) degrees (°)
(ii) radians (rad)

Angle, θ (in rad) = $\frac{s}{r}$

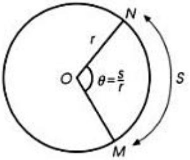
For the angle of a complete revolution 360°,

$s = 2\pi r$

$\therefore \theta = \frac{2\pi r}{r} = 2\pi$

$\therefore 2\,\pi$ rad = 360°

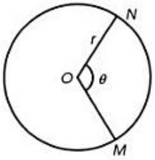
π rad = 180°



2. Arc length and area of sector

From $\theta = \frac{s}{r}$, $s = r\,\theta$.

Area of sector MON = $\frac{1}{2}r^2\theta$, where θ is in rad



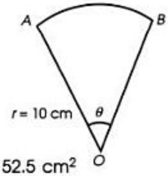
Example 1

The diagram shows a sector OAB of radius 10 cm and $\theta = 60$. Find

(i) θ in rad,

(ii) arc length AB,

(iii) area of sector AOB.



Solution (i) $60° = 60 \times \frac{\pi}{180}$ rad = 1.05 rad

(ii) $AB = r\theta = 10(1.05) = 10.5$ cm

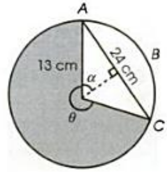
(iii) Area of sector $AOB = \frac{1}{2}r^2\theta = \frac{1}{2}(10^2)(1.05) = 52.5\text{ cm}^2$

Example 2

The diagram shows a circle of radius 13 cm and chord AC of length 24 cm. Calculate

(i) the length of the arc ABC,

(ii) the area of the shaded region. (C)



Solution $\alpha = \sin^{-1}\dfrac{12}{13} = 67.38°$

(i) Arc $ABC = r\theta = 13 \times \left[2(67.38°) \times \dfrac{\pi}{180°}\right]$

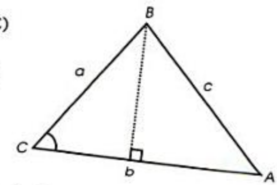
$= 30.6$ cm

(ii) Area of the shaded region $= \dfrac{1}{2}r^2\theta$

$= \dfrac{1}{2}(13)^2\left[2\pi - 2(67.38°) \times \dfrac{\pi}{180°}\right]$

$= 332$ cm^2

3. Area of triangle



Area of $\Delta ABC = \dfrac{1}{2} \times \text{base} \times \text{height} = \dfrac{1}{2}b(a \sin C)$

$= \dfrac{1}{2}ab \sin C$

Expressing height in terms of sin A and sin B,

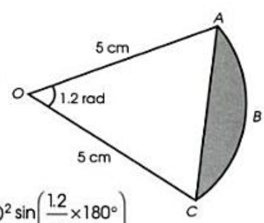
Area of $\Delta ABC = \dfrac{1}{2}ab \sin C = \dfrac{1}{2}bc \sin A = \dfrac{1}{2}ca \sin B$

Example 3

The diagram shows part of a circle, its centre O and its radius of 5 cm. Given that $\angle AOC$ is 1.2 radians, calculate

(i) the length of the arc ABC,

(ii) the area of the shaded segment. (C)



Solution (i) The length of the arc $ABC = r\theta$

$= 5(1.2) = 6$ cm

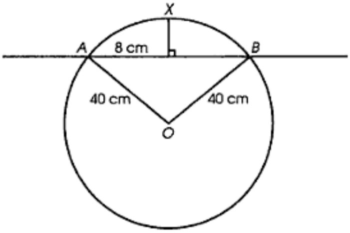
(ii) Area of the sector OAC is $\dfrac{1}{2}r^2\theta$

$= \dfrac{1}{2}(5)^2(1.2)$

Area of ΔDAC is $\dfrac{1}{2}ab\sin C = \dfrac{1}{2}(5)^2 \sin\left(\dfrac{1.2}{\pi} \times 180°\right)$

$\therefore$ area of shaded segment is $\dfrac{1}{2}(5)^2\left[1.2 - \sin\left(\dfrac{1.2}{\pi} \times 180°\right)\right] = 3.35$ cm^2

Example 4



The figure shows the circular cross-section of an uniform log of radius 40 cm floating in water. The points A and B are on the surface and the highest point X is 8 cm above the surface.
Show that $\angle AOB$ is approximately 1.29 radians.
Calculate
(i) the length of the arc AXB,
(ii) the area of the cross-section below the surface,
(iii) the percentage of the volume of the log below the surface. (C)

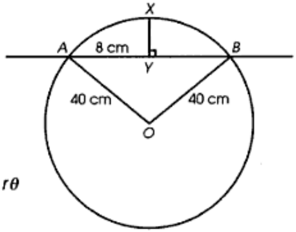
Solution Let Y be the point of intersection of AB and OX.

$OY = 40 - 8 = 32$ cm

$\cos \angle BOY = \dfrac{32}{40}$

$\therefore \angle AOB = 2\cos^{-1}\dfrac{32}{40} = 73.74°$

$= 73.74° \times \dfrac{\pi}{180°} = 1.287 = 1.29$ rad



(i) The length of the arc $AXB = r\theta$
$= 40\ (1.29) = 51.5$ cm

(ii) Area of sector $AOB = \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(40)^2(1.29) = 1029.6 \text{ cm}^2$

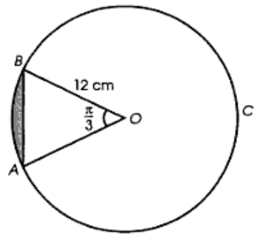
Area of triangle AOB is $\dfrac{1}{2}ab\sin C = \dfrac{1}{2}40 \times 40 \sin 73.74° = 768.0 \text{ cm}^2$

Area of cross-section below the surface $= \pi r^2 - 1029.6 + 768.0$
$= 4765 \text{ cm}^2$

(iii) Let l be the length of the log.
Volume of the log $= \pi r^2 l$
$= \pi(40)^2 l = 5027l \text{ cm}^3$
Volume of cross-section below the surface $= 4765l$
Percentage of volume of cross-section below the surface
$= \dfrac{4765l}{5027l} \times 100\% = 94.8\%$

Revision Exercises

1.

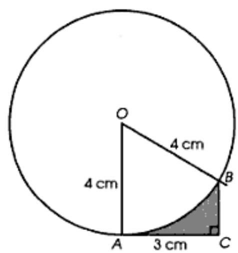


The figure shows a circle, its centre O, its radius of 12 cm and a chord AB such that angle $AOB = \dfrac{\pi}{3}$ radians. Calculate

(i) the length of the major arc ACB,

(ii) the area of the shaded region.

2.

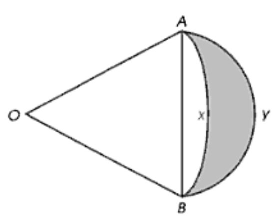


The figure shows a circle of radius 4 cm with centre O. Its radii OA and OB form a trapezium with a point C. Given that $AC = 3$ cm, calculate

(i) angle AOB in degrees,

(ii) the area of the shaded region.

3.



In the diagram, AXB is an arc of a circle, centre O, of radius 10 cm; AYB is a semicircle with AB as diameter. If triangle AOB is equilateral, find the angle AOB.

Hence calculate

(i) the length of the chord AB,

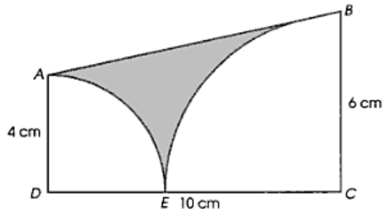
(ii) the length of the arc AXB,

(iii) the area of the sector $OAXB$,

(iv) the perimeter of the shaded region,

(v) the area of the shaded region.

4.



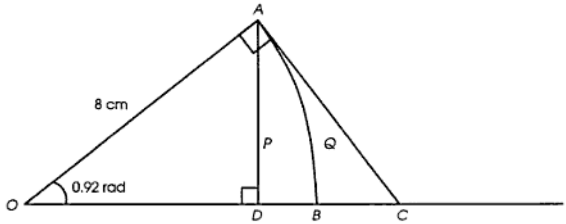
The trapezium $ABCD$ has side $CD = 10$ cm. Two arcs, with centres at D and C and radii 4 cm and 6 cm respectively meet CD at point E. Find

(i) the area of the trapezium,

(ii) the length AB,

(iii) the area of the shaded region.

5.

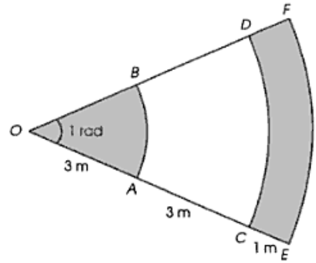


In the diagram, OAB is a sector of a circle, centre O, of radius 8 cm and angle $AOB = 0.92$ radians. The line AD is the perpendicular from A to OB. The line AC is perpendicular to OA and meets OB produced at C. Find

(i) the perimeter of the region ADB, marked P,

(ii) the area of the region ABC, marked Q. (C)

6.



AB, CD and EF are arcs of concentric circles, centre O, where angle AOB = 1 rad, $OA = AC = 3$ m and $CE = 1$ m. Calculate

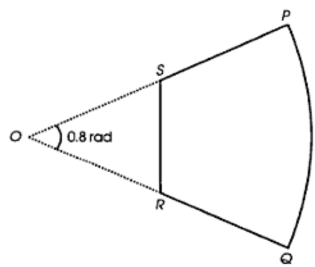
(i) the perimeter $ABDC$,

(ii) the shaded area.

7. A piece of wire is bent to form the perimeter of part of a sector with centre O and radius 10 cm. S and R are midpoints of OP and OQ respectively. RS is a straight line. Given that angle POQ = 0.8 rad, calculate

(i) the perimeter of $PQRS$,

(ii) the area enclosed by the wire.



Chapter 10

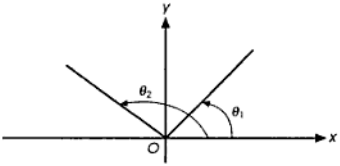
Trigonometry

Curriculum Objectives:

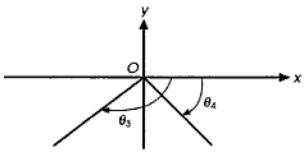
- Know the six trigonometric functions of angles of any magnitude (sine, cosine, tangent, secant, cosecant, cotangent)
- Understand amplitude and periodicity and the relationship between graphs of e.g. $\sin x$ and $\sin 2x$
- Draw and use the graphs of $y = a\sin(bx) + c$, $y = a\cos(bx) + c$, $y = a\tan(bx) + c$, where a, b are positive integers and c is an interger
- Know and use the relationships $\dfrac{\sin A}{\cos A} = \tan A$, $\dfrac{\cos A}{\sin A} = \cot A$, $\sin^2 A + \cos^2 A = 1$, $\sec^2 A = 1 + \tan^2 A$, $\operatorname{cosec}^2 A = 1 + \cot^2 A$, and solve simple trigonometric equations involving the six trigonometric functions and the above relationships (not including general solution of trigonometric equations)
- Prove simple trigonometric identities.

1. General angles

Angles measured from the x-axis, in the anticlockwise direction, are positive. For example, θ_1 and θ_2 are positive angles.--



Angles measured from the x-axis, in the clockwise direction, are negative. For example θ_3 and θ_4 are negative angles.



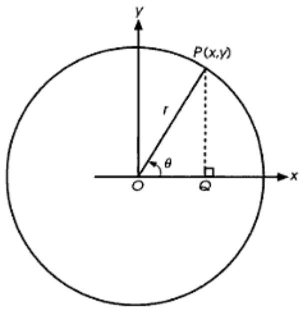
2. Trigonometrical Ratios

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{PQ}{OP} = \frac{y}{r}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{OQ}{OP} = \frac{x}{r}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{PQ}{OQ} = \frac{y}{x}$$

where x and y are coordinates of point P and r is the distance of P from O, $r^2 = x^2 + y^2$ and $x \neq 0$



3. Signs of Trigonometrical Ratios

Second quadrant, $x < 0$, $y > 0$ $\sin\theta > 0$ $\cos\theta < 0$ $\tan\theta < 0$ ONLY SINE IS POSITIVE	First quadrant, $x > 0$, $y > 0$ $\sin\theta > 0$ $\cos\theta > 0$ $\tan\theta > 0$ ALL ARE POSITIVE
Third quadrant, $x < 0$, $y < 0$ $\sin\theta > 0$ $\cos\theta < 0$ $\tan\theta > 0$ ONLY TANGENT IS POSITIVE	Fourth quadrant, $x > 0$, $y < 0$ $\sin\theta < 0$ $\cos\theta > 0$ $\tan\theta < 0$ ONLY COSINE IS POSITIVE

Example 1

Find all the angles between 0° and 360° which satisfy the equation $\sin x = -\sin 60°$. (C)

Solution Since $\sin x < 0$, x lies in the third or fourth quadrant.

$x = 180° + 60° = 240°$ or $x = 360° - 60° = 300°$

Example 2

Given that $\sin x = b$ and $0° < x < 90°$ find an expression in terms of b for
(i) $\tan x$, (ii) $\cos(-x)$.

Solution (i) Given that x is in the first quadrant,

let $\sin x = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{b}{1}$ and the reference triangle be as shown.

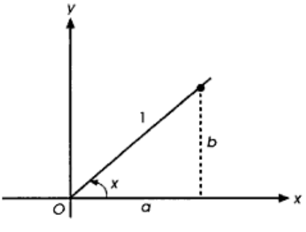
$a^2 + b^2 = 1$

$\therefore\ a = \sqrt{1-b^2}$ ($a > 0$, since x is in first quadrant)

Referring to the reference triangle,

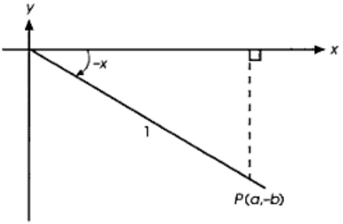
$$\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

$$= \frac{b}{\sqrt{1-b^2}}$$



(ii) $(-x)$ is measured from the x-axis in the clockwise direction as shown in the figure.

$$\cos(-x) = \frac{a}{1} = \sqrt{1-b^2}$$



4. Trigonometrical Ratios of Special angles: 30°, 45° and 60°

In a right-angled triangle OPQ,

$\angle OPQ = 30°$

$$\sin 30° = \frac{1}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

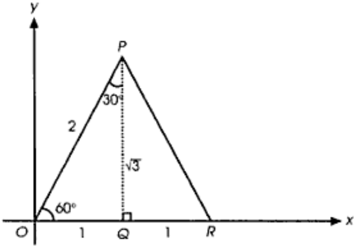
$$\tan 30° = \frac{1}{\sqrt{3}}$$

$\angle POQ = 60°$

$$\sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{1}{2}$$

$$\tan 60° = \sqrt{3}$$

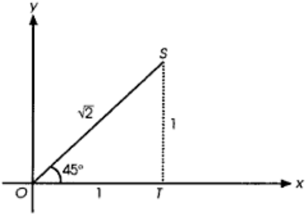


$\angle SOT = \angle OST = 45°$

$$\sin 45° = \frac{1}{\sqrt{2}}$$

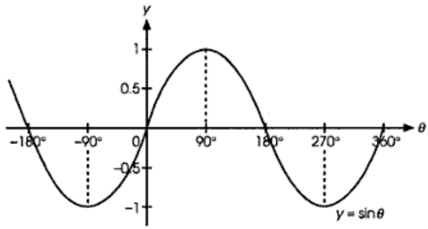
$$\cos 45° = \frac{1}{\sqrt{2}}$$

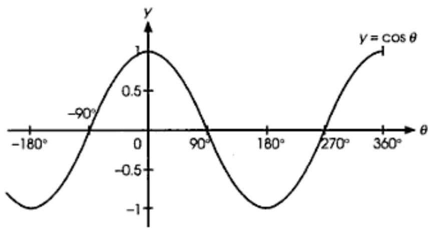
$$\tan 45° = 1$$

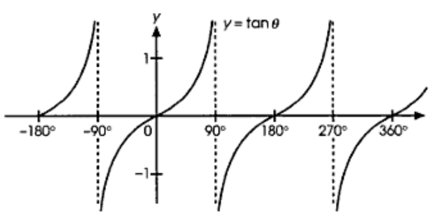


Note: You are expected to recall the trigonometrical ratios of special angles. The figures will help you remember the ratios.

5. Graphs of sinθ, cosθ and tanθ







Comparing the three graphs:

graph of sinθ	graph of cosθ	graph of tanθ
Period is 360°	Period is 360°	Period is 180°
$-1 \leq \sin\theta \leq 1$	$-1 \leq \cos\theta \leq 1$	$-\infty \leq \tan\theta \leq \infty$
Max. = 1, min. = –1 Amplitude = 1	Max. = 1, min. = –1 Amplitude = 1	No max. or min.
$\sin\theta = 0$ when $\theta = \ldots -360°$, $-180°, 0°, 180°, 360° \ldots$	$\cos\theta = 0$ when $\theta = \ldots -270°$, $-90°, 90°, 270° \ldots$	$\tan\theta = 0$ when $\theta = \ldots -360°$, $-180°, 0°, 180°, 360° \ldots$
The sine graph is symmetrical about the origin, i.e., $\sin(-\theta) = -\sin\theta$	The cosine graph is symmetrical about the vertical axis, i.e., $\cos(-\theta) = \cos\theta$	The tangent graph is symmetrical about the origin, i.e., $\tan(-\theta) = -\tan\theta$

		The vertical lines through $\theta = \ldots -270°, -90°, 90°, 270° \ldots$ where $\tan\theta$ is undefined are asymptotes of the graph

Note: The graph of $\sin\theta$ is simply the graph of $\cos\theta$ shifted 90° to the right.

Example 3

Find all the angles between 0° and 360° which satisfy the equation, $\tan 2y = -1$.
(C)

Solution $0° \le y \le 360° \Rightarrow 0° \le 2y \le 720°$ and since $\tan 2y < 0$, $2y$ lies in the second and fourth quadrants,

$\tan 2y = -\tan 45°$

$2y = 180° - 45°, 360° - 45°, 360° + 180° - 45°, 360° + 360° - 45°.$

$\quad = 135°, 315°, 495°, 675°$

$\therefore\ y = 67.5°, 157.5°, 247.5°, 337.5°$

6. Sketch Trigonometrical Graphs

Points to consider when sketching graphs:

(i) shapes of curves for general trigonometrical functions,

(ii) important points such as maximum and minimum points (which help in determining the amplitude) and points where the curve cuts the axes,

(iii) the period of the function,

(iv) the asymptotes in the case of graphs involving the tangent function.

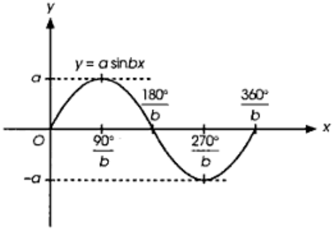
The general shapes of the following functions are shown below.

(i) $y = a \sin bx$,

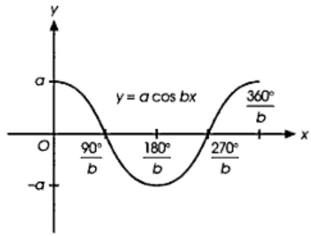
(ii) $y = a \cos bx$,

(ii) $y = a \tan bx$.

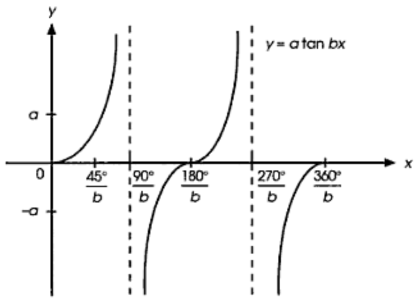
(i) $y = a \sin bx$ where a and b are constants.



(ii) $y = a \cos bx$



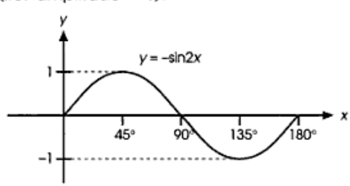
(iii) $y = a \tan bx$



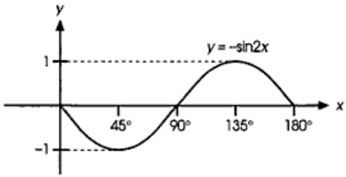
Example 4

Sketch the graph of $y = 3 - \sin 2x$ for $0° \leq x \leq 180°$. (C)

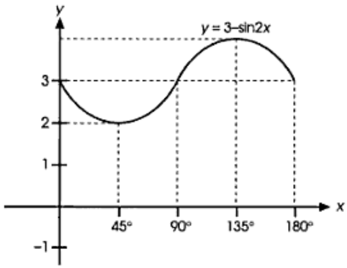
Solution $\sin 2x$ has a period of $\frac{360°}{2} = 180°$ and maximum and minimum values of 1 and –1 (i.e. amplitude = 1).



Recall $y = a \sin bx$, here $a = 1$ and $b = 2$. Hence the curve cuts the x-axis at $\frac{180°}{2} = 90°$; $\frac{360°}{2} = 180°$.



The graph of $y = -\sin 2x$ is obtained by reflecting the graph of $y = \sin 2x$ about the x-axis.



Finally, shift the graph of $y = -\sin 2x$ upward by 3 units to get the graph of $y = 3 - \sin 2x$.

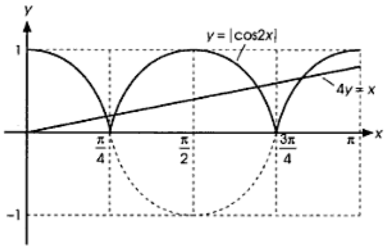
Alternatively, one can get the graph of $y = 3 - \sin 2x$ by plotting the graph at $x = 0°, 45°, 90°, 135°, 180°$ directly.

Example 5

Sketch on the same diagram the graphs of $y = |\cos 2x|$ and $4y = x$ for the domain $0 \le x \le \pi$. (C)

Solution $y = |\cos 2x|$ graph has a period $\frac{360°}{2} = \frac{2\pi}{2} = \pi$ and an amplitude of 1.

The straight line graph of $y = \frac{x}{4}$ has a gradient of $\frac{1}{4}$.



To get the graph of $y = |\cos 2x|$, first sketch the graph $y = \cos 2x$ and then reflect the part of the graph below the x-axis about the x-axis.

Note: There are four points of intersection of the two graphs, hence there are four solutions to the equation $4|\cos 2x| = x$.

7. Trigonometrical Ratios of Cotangent, Secant and Cosecant

$\tan A = \dfrac{\sin A}{\cos A}$ cotangent$A$ or $\cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}$

secant$A$ or $\sec A = \dfrac{1}{\cos A}$ cosecant$A$ or $\operatorname{cosec} A = \dfrac{1}{\sin A}$

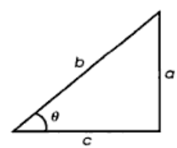
Note: signs of cotangent, secant and cosecant in the four quadrants follow those of their reciprocals, i.e. tangent, cosine and sine respectively.

Second quadrant Only sine and cosecant positive	**First quadrant** All positive
Third quadrant Only tangent and cotangent positive	**Fourth quadrant** Only cosine and secant positive

8. Fundamental Identities

Consider the right-angled triangle in the figure.

By Pythagoras' theorem, $b^2 = a^2 + c^2$ ....(1)



(1) ÷ b^2 $\quad 1 = \dfrac{a^2}{b^2} + \dfrac{c^2}{b^2}$

$1 = \sin^2\theta + \cos^2\theta$

(1) ÷ a^2 $\quad \dfrac{b^2}{a^2} = 1 + \dfrac{c^2}{a^2}$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(1) ÷ c^2 $\quad \dfrac{b^2}{c^2} = \dfrac{a^2}{c^2} + 1$

$\sec^2\theta = \tan^2\theta + 1$

In summary,

$$\boxed{\begin{array}{l} \sin^2\theta + \cos^2\theta = 1 \\ 1 + \cot^2\theta = \operatorname{cosec}^2\theta \\ \tan^2\theta + 1 = \sec^2\theta \end{array}}$$

as given in the formula list.

Example 6

Find all the angles between 0° and 360° which satisfy each of the equations.

(i) $10 \sin x \cos x = \cos x$,

(ii) $5 \tan^2 y = 5 \tan y + 3 \sec^2 y$,

(iii) $\sec\left(\frac{1}{2}z + 107°\right) = -2$ (C)

Solution (i) $10 \sin x \cos x = \cos x$

$\cos x\,(10 \sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{10}$

$x = 90°, 270°$ or $x = 5.7°, 180° - 5.7°$

$= 5.7°, 174.3°$

$\therefore\ x = 5.7°, 90°, 174.3°$ or $270°$

(ii) $5 \tan^2 y = 5 \tan y + 3 \sec^2 y$

$5 \tan^2 y = 5 \tan y + 3 (\tan^2 y + 1)$ since $\tan^2\theta + 1 = \sec^2\theta$

$2 \tan^2 y - 5 \tan y - 3 = 0$

$(2 \tan y + 1)(\tan y - 3) = 0$

$\therefore\ \tan y = -\frac{1}{2}$ or $\tan y = 3$

$\tan y = -\tan 26.6°$ or $\tan y = \tan 71.6°$

$y = 180° - 26.6°, 360° - 26.6°$ or $y = 71.6°, 180° + 71.6°$

$\therefore\ y = 71.6°, 153.4°, 251.6°, 333.4°$

(iii) $\sec\left(\frac{1}{2}z + 107°\right) = -2$

$$\frac{1}{\cos\left(\frac{1}{2}z + 107°\right)} = -2$$

$1 = -2 \cos\left(\frac{1}{2}z + 107°\right)$

$\cos\left(\frac{1}{2}z + 107°\right) = -\frac{1}{2}$

$\cos\left(\frac{1}{2}z + 107°\right) = -\cos 60°$

$\left(\frac{1}{2}z + 107°\right) = 180° - 60°, 180° + 60°, 540° - 60° = 120°, 240°, 480°$

$z = 2\,(120° - 107°), 2\,(240° - 107°), 2(480° - 107°)$

$= 26°, 266°, 746°$

$z = 746°$ is outside the required range,

$\therefore\ z = 26°, 266°$

Alternatively, consider the range of $\left(\frac{1}{2}z+107°\right)$ before giving the values of $\left(\frac{1}{2}z+107°\right)$.

$0° \le z \le 360°$

$0° \le \frac{1}{2}z \le 180°$

$0° \le \frac{1}{2}z + 107° \le 287°$

$\cos\left(\frac{1}{2}z+107°\right) = -\cos 60°$

$\therefore\ \frac{1}{2}z + 107° = 120°, 240°$

$z = 2\,(120° - 107°), 2\,(240° - 107°)$

$z = 26°, 266°$

Example 7

Show that $(\operatorname{cosec}x - 1)(\operatorname{cosec}x + 1)(\sec x - 1)(\sec x + 1) = 1$ (C)

Solution

$$\begin{aligned}
\text{LHS} &= (\operatorname{cosec}x - 1)(\operatorname{cosec}x + 1)(\sec x - 1)(\sec x + 1)\\
&= (\operatorname{cosec}^2x - 1)(\sec^2x - 1)\\
&= \operatorname{cosec}^2x \sec^2x - \sec^2x - \operatorname{cosec}^2x + 1\\
&= \frac{1}{\sin^2x}\frac{1}{\cos^2x} - \frac{1}{\cos^2x} - \frac{1}{\sin^2x} + 1\\
&= \frac{1}{\sin^2x\cos^2x} - \frac{\sin^2x}{\sin^2x\cos^2x} - \frac{\cos^2x}{\sin^2x\cos^2x} + 1\\
&= \frac{1-(\sin^2x+\cos^2x)}{\sin^2x\cos^2x} + 1 \qquad \text{since } \sin^2x + \cos^2x = 1\\
&= \frac{1-1}{\sin^2x\cos^2x} + 1 = 1 = \text{RHS}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\text{LHS} &= (\operatorname{cosec}x - 1)(\operatorname{cosec}x + 1)(\sec x - 1)(\sec x + 1)\\
&= (\operatorname{cosec}^2x - 1)(\sec^2x - 1)\\
&= \cot^2x \tan^2x\\
&= 1 \qquad \text{recall: } \tan^2\theta + 1 = \sec^2\theta\\
&= \text{R.H.S.} \qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta
\end{aligned}$$

Example 8

Prove the identity

$$\operatorname{cosec}A - \frac{\sin A}{1+\cos A} \equiv \cot A. \qquad \text{(C)}$$

Solution LHS $= \operatorname{cosec}A - \dfrac{\sin A}{1+\cos A}$

$$= \frac{1}{\sin A} - \frac{\sin A}{1+\cos A}$$

$$= \frac{1+\cos A - \sin^2 A}{\sin A(1+\cos A)}$$

$$= \frac{1+\cos A - (1-\cos^2 A)}{\sin A(1+\cos A)}$$

$$= \frac{\cos A + \cos^2 A}{\sin A(1+\cos A)}$$

$$= \frac{\cos A(1+\cos A)}{\sin A(1+\cos A)}$$

$$= \frac{\cos A}{\sin A}$$

$= \cot A =$ RHS

Example 9

Prove the identity $(\cot A - \tan A)\sin A \equiv 2\cos A - \sec A$ (C)

Solution LHS $= (\cot A - \tan A)\sin A.$

$$= \left(\frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}\right)\sin A$$

$$= \cos A - \frac{\sin^2 A}{\cos A}$$

$$= \cos A - \left(\frac{1-\cos^2 A}{\cos A}\right)$$

$$= \cos A - \frac{1}{\cos A} + \cos A$$

$= 2\cos A - \sec A =$ RHS

Revision Exercises

1. Given that $\cos 150° = -p$, express the following in terms of p.
   (a) $\cos 30°$,
   (b) $\sin 150°$,
   (c) $\tan(-30°)$,
   (d) $\cos 60°$.

2. Find the maximum and minimum values of
   (a) $-4 \cos\theta$,
   (b) $5 \cos(\theta - 30°)$,
   (c) $8 \sin 4\theta$,
   (d) $8 - 7\sin 2\theta$,
   (e) $6 - 2|\sin \theta|$.

3. The function $f(x) = a\sin x + b$, where $a > 0$, has a maximum value of 7 and a minimum value of -3. Find the value of a and of b. (C)

4. Sketch the following graphs:
   (a) $y = 3 - 2 \sin x$ for $0° \le x \le 360°$
   (b) $y = 3|\cos x|$ for $0° \le x \le 360°$
   (c) $y = 3 - |\tan x|$ for $0 \le x \le \frac{3}{2}\pi$
   (d) $y = 2 \sin 2x - 1$ for $0 \le x \le \pi$
   (e) $y = |\tan\frac{x}{2}| + 2$ for $0° \le x \le 720°$

5. Sketch on the same diagram, the graphs of $y = 2 \cos x$ and $y = 1 - |\sin 2x|$ for $0 \le x \le 2\pi$. Hence, state the number of solutions in the interval of the equation $2 \cos x + |\sin 2x| = 1$.

6. Solve the following equations for $0° < x < 360°$.
   (a) $\sin 2x = 0.98$
   (b) $\cos 2x + \sin 50° = 0$
   (c) $\tan 2x = -\cos 67°$
   (d) $\sin(\frac{1}{2}x - 30°) = 0.5$
   (e) $8(\tan x - 2) = 3(2 \tan x + 3)$

7. Solve the following equations for $0° \le x \le 360°$.
   (a) $\operatorname{cosec}^2 x = 3 \sin x$
   (b) $5 \cot^2 x - 23 \cot x - 10 = 0$
   (c) $2 \sec 2x + 5 = 0$
   (d) $2\sec^2 x - 5 \tan x = 5$
   (e) $2\sin x + \cos x = \dfrac{2}{\sin x + 2\cos x}$

8. Given that $\sin A = \frac{4}{5}$ and $\cos B = \frac{15}{17}$ and that A and B are acute, find, without using tables or calculators, the value of
   (a) $\sin(A + B)$,
   (b) $\cos(A + B)$,
   (c) $\tan(A - B)$.

9. Solve the following equations for $0° \le x \le 360°$.
   (a) $6 \sin(2x - 35°) = 1$
   (b) $2 \sin 2x + \cos x = 0$
   (c) $2 \cos 2x - 5 \sin x + 1 = 0$
   (d) $2 \tan x = \tan(30° - x)$

10. Prove the following identities:
   (a) $\operatorname{cosec} A \equiv \sec A \cot A$
   (b) $\dfrac{1}{\sin A \cos A} \equiv \tan A + \cot A$
   (c) $\sin A \cos A \equiv \dfrac{1 + \cos A}{\operatorname{cosec} A + \cot A + \tan A}$
   (d) $\sin A - \cos A \equiv \dfrac{\sec A - \operatorname{cosec} A}{\tan A + \cot A}$
   (e) $\operatorname{cosec} A \equiv \dfrac{\cot A - \tan A}{2\cos A - \sec A}$

11. (a) Find all values between 0° and 360° which satisfy the equation
      (i) $2 \cot 2x = 5$,
      (ii) $3 \sin y \tan y + 8 = 0$.
   (b) Show that
   $$\frac{2 - \operatorname{cosec}^2 A}{\operatorname{cosec}^2 A + 2\cot A} \equiv \frac{\sin A - \cos A}{\sin A + \cos A} \qquad \text{(C)}$$

Chapter 11

Permutations and Combinations

Curriculum Objectives:

- Recognise and distinguish between a permutation case and a combination case
- Know and use the notation $n!$ (with $0! = 1$), and the expressions for permutation and combinations of n items taken r at a time
- Answer simple problems on arrangement and selection (cases with repetition of objects, or with objects arranged in a circle or involving both permutations and combinations, are excluded).

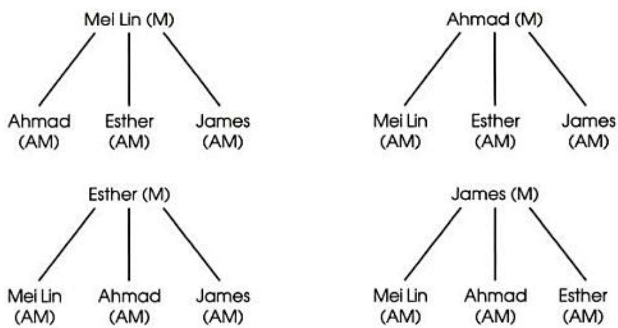
A **permutation** is an arrangement of a set of objects choosen from a given number of objects. The order of the objects in the chosen set is taken into consideration.

A **combination** is a selection of a set of objects chosen from a given number of objects. The order of the objects in the chosen set is not taken into consideration.

Example 1

Find the number of possible ways of (permutations) of appointing a monitor and an assistant monitor from 4 possible candidates, Mei Lin, Ahmad, Esther and James.

Solution



There are 12 possible ways of appointing a monitor (M) and an assistant monitor (AM). It can be seen that there are 4 possible choices of monitors. For each choice of monitor, there are 3 possible choices of assistant monitors. Hence, the number of possible ways of appointing a monitor and an assistant monitor is $4 \times 3 = 12$.

AlternativeMethod

Monitor	Assistant Monitor
Mei Lin Ahmad	Ahmad Mei Lin
Mei Lin Esther	Esther Mei Lin
Mei Lin James	James Mei Lin
Ahmad Esther	Esther Ahmad
Ahmad James	James Ahmad
Esther James	James Esther

From the second method, it can be seen that the order of appointment is taken into consideration, i.e.

Mei Lin (M) and Ahmad (AM) is different from Ahmad (M) and Mei Lin (AM).

Let's consider a case where the order is not taken into consideration.

Example 2

Consider the 4 candidates in Example 1. Find the number of ways (combinations) of selecting 2 prefects.

Solution In this case, since order is not taken into consideration, it can be seen that for every two permutations in *Example 1, Method 2*, there is one possible combination.

Monitor	Assistant Monitor
Mei Lin Ahmad	Ahmad Mei Lin
Mei Lin Esther	Esther Mei Lin
Mei Lin James	James Mei Lin
Ahmad Esther	Esther Ahmad
Ahmad James	James Ahmad
Esther James	James Esther

Table 1

Prefects
Mei Lin and Ahmad
Mei Lin and Esther
Mei Lin and James
Ahmad and Esther
Ahmad and James
Esther and James

Table 2

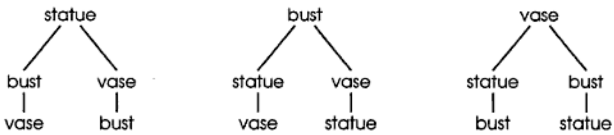
From Table 2, there are 6 combinations of prefects.

i.e. the number of combinations $= \frac{12}{2} = \frac{3 \times 4}{2} = 6$.

Example 3

An art collector wants to display a statue, a bust and a vase on 3 tables. Find the number of ways of arrangement (permutations).

Solution



There are 3 choices of display for the first table; 2 choices for the 2nd table and 1 choice for the 3rd table. The number of permutations $= 3 \times 2 \times 1 = 6$.

Note: factorial $3 = 3! = 3 \times 2 \times 1$

In general, $n! = n \times (n-1) \times (n-2) \times (n\ 3) \times \ldots$.
Factorial $n(n!)$ is the product of n and all the positive numbers less than n.

When a set of n items is chosen from n items and arranged all at a time (i.e. n items at a time), the number of permutations is $n!$.

The notation for the number of permutations of n items taken all at a time is ${}^nP_n$

where ${}^nP_n = n!$

Note: By definition $0! = 1$

In Example 1 when we consider the number of permutations of a set 2 'items' from 4 'items', we get $4 \times 3 = 12$.

Since $4! = 4 \times 3 \times 2 \times 1$,

$$4 \times 3 = \frac{4!}{2!} = \frac{4!}{(4-2)!}$$

When a set of r items is chosen from n items, the number of different permutations (or arrangements) is $\frac{n!}{(n-r)!}$.

The notation for the number of permutations of n items taken r at a time is ${}^nP_r$,

where ${}^nP_r = \frac{n!}{(n-r)!}$.

Product Principle:
When a first process can be done in a number of ways, a second process in b number of ways, a third process in c number of ways and so on, then the number of ways in which all the processes can be done is $a \times b \times c \times \ldots$.

Example 6

There are 4 basic steps to making a teddy bear: choosing and cutting the fabric; sewing the pieces; stuffing the bear and sewing on the facial expression.

The variations in each step are:
1. There are 5 types of furry fabric;
2. 2 methods of sewing;
3. 3 types of stuffing;
4. 4 facial expressions.

Find the number of ways all 4 basic steps can be carried out.

Solution The number of ways a teddy bear can be made is $5 \times 2 \times 3 \times 4 = 120$.

Revision Exercises

1. 13 cards of diamond from a deck of playing cards are to be arranged in a row. How many permutations for doing it are there?

2. Find the number of ways 5 runners can be arranged at the starting point of a 6-lane track.

3. Find the number of permutations in which a car dealer can display his 5 cars: a Mercedes, a Volkswagen, a BMW, a Toyota and a Rolls Royce when
   (i) he can display them in any order;
   (ii) the Rolls Royce must be in the first parking lot and the Mercedes in the last lot.

4. 3 girls and 4 boys have to arranged and seated on 7 chairs. Find the number of arrangements when
   (i) they can be seated in any order;
   (ii) they are to be seated in a boy-girl-boy-girl-boy sequence.

5. Find the number of ways 6 bottles can be chosen from 60 bottles.

6. Find the number of ways of selecting cards consisting 1 spade, 2 diamonds, 3 hearts and 4 clubs from a pack of 52 cards.

7. There are 6 blue balls, 7 yellow balls and 8 red balls. Find the number of combinations of a set of 1 blue ball, 7 yellow balls and 2 red balls.

8. A baker wants to bake some buns. Firstly, he can choose to use butter, margarine or oil. Secondly, he may use either plain flour or bread flour. Thirdly, he has a choice of castor white sugar or brown sugar. Lastly, he can choose to use luncheon meat, sausages, ham or tuna as filing. Find the number of ways the baker can bake a batch of buns when he can only choose 1 type of ingreadient at each stage.

9. In a maze of tunnels, there are 3 tunnels leading from point A to point B, 6 tunnels leading from B to C and 4 tunnels leading from point C to point D. Find the number of routes a hamster can take to move from point A to point D.

10. A man with a hearty appetite wants to have 4 servings of food. He can choose stalls which sell nasi lemak, laksa, fish ball noodles, roti prata, pizza, mee goreng and dumpling soup.
    (i) How many ways can the man choose 4 different servings of food?
    (ii) Find the number of permutations in which the man may eat his 4 different servings of food.
    (iii) Find the number of permutations in which the man may eat his 4 different servings of food, given the condition that he must start with laksa.

Chapter 12

Binomial Expansion

Curriculum Objectives:

- Use the Binomial Theorem for expansion of $(a+b)^n$ for positive integral n
- Know and use the general term ${}^nC_rA^{n-r}b^r, 0 < r \leqslant n$ (knowledge of the greatest term and properties of the coefficients is not required).

A binomial is an expression with two terms in it, for example, $(a+b)$, $(x+y)$, $(4a+b^2)$, and $(2-3x^2)$. Consider the first three expansions of binomial $(a+b)$.

$(a+b) = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

For $(a+b)^3$,

Term	Coefficient
a^3	1
a^2b	3
ab^2	3
b^3	1

In general, for a positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where coefficient $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-r)!r!}$.

The coefficient can also be expressed as ${}^nC_r$.

Note: The $(r+1)^{\text{th}}$ term $= {}^nC_ra^{n-r}b^r$ is called the **general term**. a or b can be terms like $\pm px^q$, $\pm\dfrac{p}{q}x$ or even $(px \pm q)$ where p and q are constants.

Example 1

In the expansion of $(k+x)^8$, where k is a positive constant, the coefficients of x^2 and x^3 are equal. Find the value of k. (C)

Solution Recall that the general term is ${}^nC_ra^{n-r}b^r$.

$\therefore$ General term of $(k+x)^8$ is ${}^8C_rk^{8-r}x^r$.

The term with x^2 is the third term when $r = 2$, i.e., ${}^8C_2k^{8-2}x^2 = {}^8C_2k^6x^2$.

The term with x^3 is the fourth term when $r = 3$, i.e., ${}^8C_3k^{8-3}x^3 = {}^8C_3k^5x^3$.

Equating the coefficients of the two terms,

$^8C_2k^6 = {}^8C_3k^5$

$28\ k = 56$

$\therefore\ k = 2$

Example 2

Write down and simplify the expansion of $(2 - p)^5$. Use this result to find the expansion of $\left(2-2x+\frac{x}{2}\right)^5$ in ascending powers of x as far as the term in x^2. (C)

Solution $(2 - p)^5 = 2^5 + {}^5C_1 2^4(-p) + {}^5C_2 2^3(-p)^2 + {}^5C_3 2^2(-p)^3 + {}^5C_4 2(-p)^4 + (-p)^5$

$= 32 - 80p + 80p^2 - 40p^3 + 10p^4 - p^5$

To find the expansion of $\left(2-2x+\frac{x}{2}\right)^5$, substitute $p = 2x-\frac{x}{2}$.

$$\left(2-2x+\frac{x}{2}\right)^5 = 32-80\left(2x-\frac{x}{2}\right)+80\left(2x-\frac{x}{2}\right)^2-40\left(2x-\frac{x}{2}\right)^3$$

$$+10\left(2x-\frac{x}{2}\right)^4-\left(2x-\frac{x}{2}\right)^5$$

$$= 32-160x+40x+80\left(4x^2-2x^2+\frac{x^2}{4}\right)\ldots$$

$$= 32-120x+180x^2\ldots$$

There is no need to consider the other terms since the question asked for terms as far as the term in x^2.

Example 3

Find, in its simplest form, the coefficient of x^4 in the expansion of

(i) $(1 + 3x)^6$

(ii) $\left(x^2+\frac{3}{x}\right)^5$. (C)

Solution (i) For $(1 + 3x)^6$, the term with x^4 is the fifth term when $r = 4$, i.e.

$^6C_4 1^{6-4}(3x)^4 = {}^6C_4 1(3x)^4$.

Coefficient of x^4 is $^6C_4(81) = 1215$

(ii) Both terms in $\left(x^2+\frac{3}{x}\right)^5$ are expressions in x.

The general term in the expansion of $\left(x^2+\frac{3}{x}\right)^5$ is

$^5C_r(x^2)^{5-r}\left(\frac{3}{x}\right)^r = {}^5C_r x^{10-2r}\frac{3^r}{x^r} = {}^5C_r 3^r x^{10-3r}$.

When $10 - 3r = 4$

$r = 2$

The coefficient of x^4 is $^5C_2(3^2) = 90$.

Revision Exercises

1. Expand (a) $(1+2x)^3$ (b) $\left(3+\frac{1}{x}\right)^4$ (c) $\left(2x-\frac{y}{3}\right)^4$

2. Find, in descending powers of x, the expansion of
   (a) $(2-x)^5$
   (b) $\left(2x-\frac{1}{2}\right)^4$
   (c) $\left(\frac{x}{2}+x^2\right)^4$

3. Find, in ascending powers of x, the expansion of
   (a) $(b+2x)^5$
   (b) $\left(1-\frac{x^2}{4}\right)^3$
   (c) $\left(x-\frac{2}{x}\right)^4$

4. Find the first 4 terms in the expansion of $\left(2-\frac{x}{2}\right)^5$ in descending powers of x. Hence, or otherwise, find the coefficient of x^6 in the expansion $(1+2x)^2\left(2-\frac{x}{2}\right)^5$.

5. Find, in ascending powers of x, the first 3 terms in the expansion of $(x^2-3x+2)^4$.

6. Write down and simplify, in ascending powers of x, the first three terms of the expansions
   (a) $\left(3+\frac{x}{2}\right)^4$
   (b) $(1-2x)^4$.

   Hence, or otherwise, find the coefficient of x^2 in the expansion of $\left(3-\frac{11}{2}x-x^2\right)^4$.

7. If the first 3 terms (in ascending powers of x) in the expansion of $\left(2-\frac{3}{4}x\right)^n$ is $16-48+54$, find the values of x and n.

8. (a) Evaluate the coefficient of x^9 in the expansion of $(1+2x)(3+x)^{11}$.
   (b) Evaluate the coefficient of x^5 in the expansion of $\left(x^2-\frac{2}{x}\right)^7$.

(c) The first three terms in the binomial expansion of $(a+b)^n$, in ascending powers of b, are denoted by p, q and r respectively. Show that

$$\frac{q^2}{pr}=\frac{2n}{n-1}.$$

Given that $p = 4$, $q = 32$ and $r = 96$, evaluate n. (C)

9. The coefficient of x^3 in the expansion of $(2 + ax)(1 - 3x)^6$ is 405. Find the value of a.

10. (a) Find the coefficient of x in the expansion of $\left(x^2-\frac{3}{x}\right)^5$.

(b) Obtain the first 4 terms in the expansion of $(1+ p)^7$ in ascending powers of p. Hence find the coefficient of x^3 in the expansion of $(1 + x + 2x^2)^7$. (C)

Chapter 13

Vectors in Two Dimensions

Curriculum Objectives:

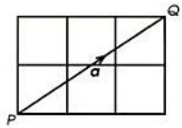
- Use vectors in any form, e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$, $\overrightarrow{\mathrm{AB}}$, $\mathbf{p}$, $a\mathbf{i}+b\mathbf{j}$
- Know and use position vectors and unit vectors
- Find the magnitude of a vector. Add and subtract vectors and multiply vectors by scalars
- Compose and resolve velocities
- Use relative velocity including solving problems on interception (but not closest approach).

Scalar Quantity: A quantity which has magnitude but no direction for example, distance, speed, time and mass.

Vector Quantity: A quantity which has both magnitude and direction for example, displacement, velocity, acceleration and forces.

1. Notation of Vector

There are a few ways of representing a vector PQ (or vector a):



(i) $\mathbf{PQ}$ or $\overrightarrow{PQ}$ where the length of the line, i.e. $|\mathbf{PQ}|$ or $|\overrightarrow{PQ}|$ or PQ gives the magnitude and the direction of the vector is from P to Q.

(ii) $\boldsymbol{a}$ or $\underset{\sim}{a}$ where the magnitude of the vector is $|\boldsymbol{a}|$ or $|\underset{\sim}{a}|$.

(iii) column matrix $\begin{pmatrix} u \\ v \end{pmatrix}$ where $u > 0$ is the movement horizontally to the right and $v > 0$ is the vertical movement to the top.

For example,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Magnitude of $\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{u^2 + v^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$

In this chapter, we will denote vectors by bold alphabets (***PQ*** or ***a***) or as column matrix $\begin{pmatrix} u \\ v \end{pmatrix}$ or with arrows over the alphabets ($\overrightarrow{PQ}$).

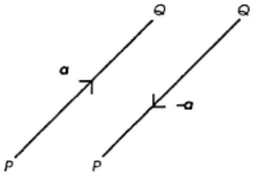
2. Equal Vector

When two vectors ***a*** and ***b*** have the same magnitude and direction, they are equal, i.e. ***a*** = ***b***

3. Zero Vector

A vector with zero magnitude and no direction is called a **zero** or **null vector**, i.e. 0 or **0**.

4. Negative Vector



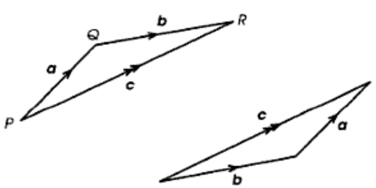
When a vector ***QP*** has the same length but is opposite in direction to ***PQ***, we say that ***QP*** is the **negative vector** of ***PQ***, i.e.

QP = −***PQ*** = −***a***

5. Addition of Vectors

PQ + ***QR*** = ***PR***

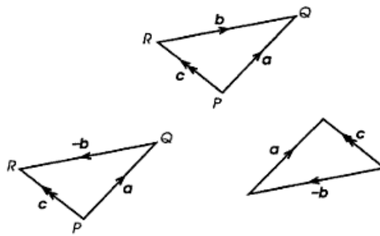
or ***a*** + ***b*** = ***b*** + ***a*** = ***c***



6. Subtraction of Vectors

$$PQ - QR = PQ + (-QR)$$
$$= PR$$

or $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$
$= (-\boldsymbol{b}) + \boldsymbol{a}$
$= \boldsymbol{c}$



7. Multiplication of a Vector by a Scalar

When a vector $\boldsymbol{a}$ is multiplied by a scalar k, we get $k\boldsymbol{a}$.
If $k > 0$, then $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$.
If $k < 0$, then $k\boldsymbol{a}$ is in the opposite direction.
If $k = 0$, then $k\boldsymbol{a}$ is a zero vector.
If $k \neq 0$, but $\boldsymbol{a} = 0$, i.e. $\boldsymbol{a}$ is a zero vector, then $k\boldsymbol{a} = 0$, i.e. $k\boldsymbol{a}$ is also a zero vector.
We express multiplication of column vector as:

When $\boldsymbol{a} = \begin{pmatrix} u \\ v \end{pmatrix}$,

$$k\boldsymbol{a} = k \times \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} ku \\ kv \end{pmatrix}$$

Properties of Scalar Multiplication

1. $(mn)\boldsymbol{a} = m(n\boldsymbol{a})$
2. $(m+n)\boldsymbol{a} = m\boldsymbol{a} + n\boldsymbol{a}$

8. Parallel Vectors

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if they are in the same ($k > 0$) or opposite ($k < 0$) direction. $\boldsymbol{a} /\!/ \boldsymbol{b} \Leftrightarrow \boldsymbol{a} = k\boldsymbol{b}$ where k is a scalar, $\boldsymbol{a} \neq 0$, $\boldsymbol{b} \neq 0$ and $k \neq 0$

9. Unit Vector

A vector whose magnitude is 1 is called a unit vector. For example, $\boldsymbol{a}$ has magnitude $|\boldsymbol{a}|$, unit vector, $\boldsymbol{a} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$ has the same direction as $\boldsymbol{a}$.

Using column vectors, we can write the unit vector as

$$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{u^2 + v^2}} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Comparing the coefficients of $\boldsymbol{b}$,
$p = (1 - q)$ ....(2)
Substitute (1) into (2).

$$p = 1 - \frac{3}{2}p$$

$$\frac{5}{2}p = 1 \quad \therefore \quad p = \frac{2}{5}$$

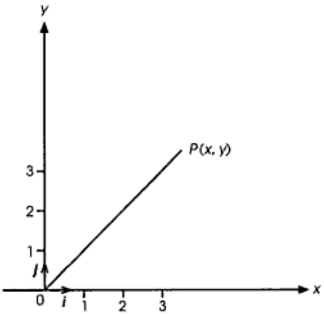
Substitute $p = \frac{2}{5}$ into (1).

$$q = \frac{3}{2}\left(\frac{2}{5}\right) = \frac{3}{5}$$

11. Two Dimensional Vectors in Cartesian Forms

Let unit vectors in the x-axis and y-axis be $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively.
A point P has Cartesian coordinates (x, y).

Position vector $\boldsymbol{OP} = \boldsymbol{p} = \begin{pmatrix} x \\ y \end{pmatrix}$



$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$|\boldsymbol{OP}| = \left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = |x\boldsymbol{i} + y\boldsymbol{j}| = \sqrt{x^2 + y^2}$$

If $\quad \boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

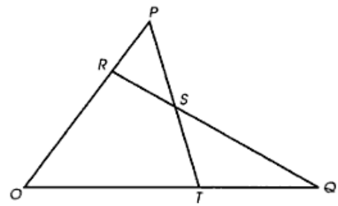
$$\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\boldsymbol{a} + \boldsymbol{b} = (x_1 + x_2)\boldsymbol{i} + (y_1 + y_2)\boldsymbol{j} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

Example 3

(a) The position vectors of A, B and C relative to an origin O are $-\boldsymbol{i} + p\boldsymbol{j}$, $5\boldsymbol{i} + 9\boldsymbol{j}$ and $6\boldsymbol{i} + 8\boldsymbol{j}$ respectively. Determine the value of p for which A, B and C are collinear. Given that D is a point on OC such that $\overrightarrow{OD}$ is a unit vector, find the position vector of D relative to O

(b)



Point P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, relative to the point O. The point R on $\overrightarrow{OP}$ is such that $\overrightarrow{OR} = \frac{2}{3}\overrightarrow{OP}$. The point S on $\overrightarrow{RQ}$ is such that $\overrightarrow{RS} = \frac{1}{3}\overrightarrow{RQ}$.

(i) Express $\overrightarrow{OS}$ and $\overrightarrow{PS}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$.

(ii) $\overrightarrow{PS}$ produced meets $\overrightarrow{OQ}$ at T, where $\overrightarrow{OT} = \lambda\overrightarrow{OQ}$. Express $\overrightarrow{PT}$ in terms of λ, $\boldsymbol{p}$ and $\boldsymbol{q}$, and evaluate λ. (C)

Solution (a) Let $\boldsymbol{OA} = \boldsymbol{a} = -\boldsymbol{i} + p\boldsymbol{j}$

$\boldsymbol{OB} = \boldsymbol{b} = 5\boldsymbol{i} + 9\boldsymbol{j}$

$\boldsymbol{OC} = \boldsymbol{c} = 6\boldsymbol{i} + 8\boldsymbol{j}$

A, B and C are collinear $\Rightarrow \boldsymbol{AB} = k\,\boldsymbol{BC}$

$\boldsymbol{AB} = \boldsymbol{b} - \boldsymbol{a} = (5\boldsymbol{i} + 9\boldsymbol{j}) - (-\boldsymbol{i} + p\boldsymbol{j}) = 6\boldsymbol{i} + (9 - p)\boldsymbol{j}$

$\boldsymbol{BC} = \boldsymbol{c} - \boldsymbol{b} = (6 - 5)\boldsymbol{i} + (8 - 9)\boldsymbol{j} = \boldsymbol{i} - \boldsymbol{j}$

$\boldsymbol{AB} = k\,\boldsymbol{BC}$

$6\boldsymbol{i} + (9 - p)\boldsymbol{j} = k(\boldsymbol{i} - \boldsymbol{j})$

Comparing coefficients of $\boldsymbol{i}$,

$k = 6$ .... (1)

Comparing coefficients of $\boldsymbol{j}$,

$9 - p = -k$............ (2)

Substitute $k = 6$ into (2).

$9 - p = -6$

$\therefore\ p = 15$

$|\boldsymbol{OC}| = \sqrt{6^2 + 8^2} = 10$

$\therefore \boldsymbol{OD} = \frac{1}{10}\boldsymbol{OC} = \frac{6}{10}\boldsymbol{i} + \frac{8}{10}\boldsymbol{j} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

(b) (i) Given $\boldsymbol{OP} = \boldsymbol{p}$, $\boldsymbol{OQ} = \boldsymbol{q}$, $\boldsymbol{OR} = \frac{2}{3}\boldsymbol{p}$ and $\boldsymbol{RS} = \frac{1}{3}\boldsymbol{RQ}$

$\boldsymbol{OS} = \boldsymbol{OR} + \boldsymbol{RS}$

$\boldsymbol{RQ} = \boldsymbol{RO} + \boldsymbol{OQ}$

$= -\boldsymbol{OR} + \boldsymbol{OQ}$

$= -\frac{2}{3}\boldsymbol{p} + \boldsymbol{q}$

$\boldsymbol{RS} = \frac{1}{3}\left(-\frac{2}{3}\boldsymbol{p} + \boldsymbol{q}\right) = -\frac{2}{9}\boldsymbol{p} + \frac{1}{3}\boldsymbol{q}$

Example 4

A car A is moving with an actual velocity of 70 km h^{-1}. What is its velocity as observed by:

(i) B, a man sitting at a bus-stop.

(ii) C, a man sitting on a bus moving at a velocity of 40 km h^{-1} toward the car, A;

(iii) D, a man sitting on a motorcycle moving at a velocity of 50 km h^{-1} away from the car, A?

Solution (i) $\underset{\circ\!\rightarrow}{A} \quad \underset{\circ}{B}$

B, the man sitting at the bus-stop is not moving relative to Earth, he observes the actual velocity of the car, A.

Velocity of A relative to B, $V_{AB} = V_A - V_B$

$= 70 - 0$

$= 70$ km h^{-1}

(ii) $\underset{\circ\!\rightarrow}{A} \quad \underset{\leftarrow\!\circ}{C}$

As C is moving towards the car A, the latter's velocity is perceived to be faster than its actual velocity by observer C.

Velocity of A relative to C, $V_{AC} = V_A - V_C$

$= 70 - (-40)$

$= 110$ km h^{-1}

Note: Velocity of A is assigned the positive sign. Since velocity of C is in the opposite direction, it is assigned the negative sign.

(iii) $\underset{\circ\!\rightarrow}{A} \quad \underset{\circ\!\rightarrow}{D}$

Velocity of the car A is observed to be slower that its actual velocity as the observer D is moving away from A.

Velocity of A relative to D, $V_{AD} = V_A - V_D$

$= 70 - (50)$

$= 20$ km h^{-1}

In general, velocity of P observed by Q, $V_{PQ} = V_P - V_Q$
where V_P is the velocity of P and V_Q is the velocity of Q, the observer.

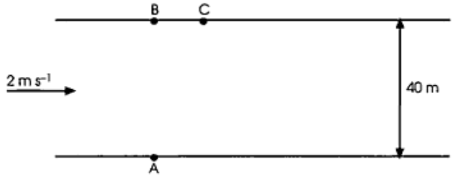
Example 5

A train P is moving at a speed of 100 km h^{-1} due east. Another train Q is moving at a speed of 80 km h^{-1} due north. Find

(i) the velocity of P relative to Q,

(ii) the velocity of Q relative to P.

to the water. But alas! He found himself at point C.

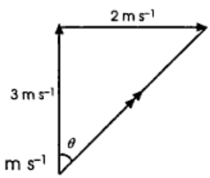


(a) (i) What was the resultant velocity of the kayak?
(ii) What was the distance between point B and point C?

(b) The man decided to try again. Given that that speed of the flow of the river and the speed of the kayak relative to the water remained the same.
(i) At what bearing should he steer the kayak in order to reach point B?
(ii) How long did he take to reach point B?

Solution (a) (i) Resultant speed of the kayak is $\sqrt{3^2+2^2} = \sqrt{13}$ m s^{-1}

$= 3.6$ m s^{-1}



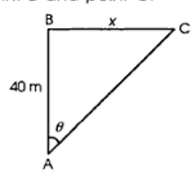
$\tan\theta = \dfrac{2}{3}$

$\theta = 33.7°$

Resultant velocity of kayak is 3.6 m s^{-1} at a bearing of 033.7°.

Note: Since relative velocity is a vector, it is necessary to give its magnitude and direction (bearing).

(ii) Let x be the distance between point B and point C.



$\tan\theta = \dfrac{x}{40}$

$x = 40 \tan 33.7°$

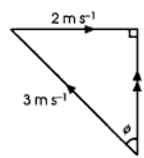
$= 26.7°$

Distance between point B and point C is 26.7 m.

(b) (i) $\sin\phi = \dfrac{2}{3}$

$\phi = 41.8°$

He should steer the kayak at a bearing of $360° - 41.8° = 318.2°$



(ii) His resultant speed is $\sqrt{3^2-2^2} = \sqrt{5}$

$= 2.2$ m s^{-1}

Time taken for him to reach point B is $\dfrac{40}{2.2} = 17.9$ s

Example 7

A robber and his accomplice have just robbed a bank and are getting away in their car, A. A police patrol car, B, gives chase. At an instant, car A is moving at a speed of 100 km h^{-1} due east. Patrol car B is 200 m due south of A and is moving at 140 km h^{-1}. Assuming that the speeds of both cars remain constant and that there are no obstacles along their paths,

(i) find the bearing that the patrol car B has to be steered in order to intercept car A;

(ii) find the time taken for the interception. (Give your answer in seconds.)

Solution (i) $\sin\theta = \dfrac{100}{140}$

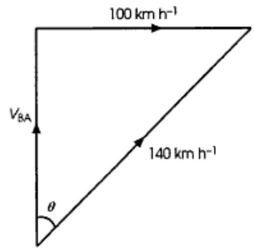
$\theta = 45.6°$

The bearing of patrol car B has to be 045.6°.

(ii) The velocity of B relative to A is V_{BA}.

$$\begin{aligned} V_{BA} &= V_B - V_A \\ &= \sqrt{140^2 - 100^2} \\ &= 98.0 \text{ km h}^{-1} \\ &= \frac{98.0 \times 1000}{3600} \\ &= 27.2 \text{ m s}^{-1} \end{aligned}$$

The time taken for the interception is $\dfrac{200}{27.2} = 7.35$ s

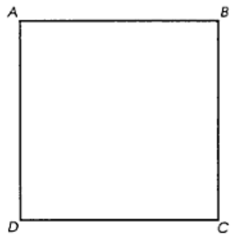


Note: Be careful when you deal with questions that use different units. In this case, km and m; h and s. Remember to convert to the same units. Since 1 km = 1000 m and 1 h = 3600 s,

$\dfrac{1 \text{ km}}{1 \text{ h}} = \dfrac{1000}{3600}$ m s^{-1}.

Revision Exercises

1.



$ABCD$ is a square where $\mathbf{AB} = \mathbf{a}$, $\mathbf{BC} = \mathbf{b}$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the following vectors:

(a) $\mathbf{AC}$ (b) $\mathbf{AD}$

(c) $\mathbf{AB} + \mathbf{AD}$ (d) $\mathbf{BD}$

(e) $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DA}$

2. Given that O, P and Q are three points such that $\mathbf{OP} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{PQ} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\mathbf{OQ} = \begin{pmatrix} p \\ q \end{pmatrix}$. Find the value of p and of q and hence $|\mathbf{OQ}|$. Show that $\angle OPQ$ is a right-angled triangle and that triangle OPQ is isosceles.

3. Simplify:

(a) $3(\mathbf{a} - \mathbf{b}) + 5(2\mathbf{a} + \mathbf{b})$

(b) $\frac{1}{2}(\mathbf{a} + 3\mathbf{b}) -2(\mathbf{b} - \mathbf{a})$

(c) $3\begin{pmatrix} 7 \\ 6 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 6\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

(d) $3(\mathbf{a} - \mathbf{b}) + 5(2\mathbf{a} + \mathbf{b})$, given that $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

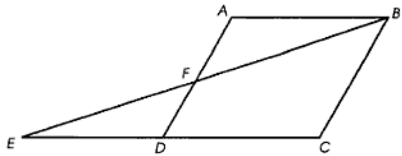
(e) $m(\mathbf{a} - \mathbf{b}) - n(2\mathbf{a} + \mathbf{b})$, given that $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $m = -2$ and $n = 1$.

4. The vector $\mathbf{OA}$ has a magnitude of 30 units and has the same direction as the vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$. Express $\mathbf{OA}$ as a column vector. The vector $\mathbf{OB}$ has a magnitude of 20 units and has the same direction as the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find the vector $\mathbf{OA} + \mathbf{OB}$.

5. The position vectors of A and B, relative to O, are $4\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} - 9\mathbf{b}$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$,

(i) the position vector of C, the mid point of $\mathbf{AB}$,

(ii) the unit vector of $\mathbf{OC}$.



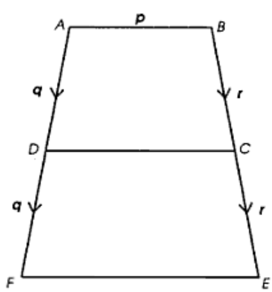
6. $ABCD$ is a parallelogram. The point E lies on CD produced such that $CE = 2CD$. F is the point of intersection of AD and BE.
   (a) Given that $\mathbf{AB} = \mathbf{p}$ and $\mathbf{BC} = \mathbf{q}$, find in terms of $\mathbf{p}$ and $\mathbf{q}$, the vectors
   (i) $\mathbf{AD}$, (ii) $\mathbf{CE}$,
   (iii) $\mathbf{BE}$, (iv) $\mathbf{AE}$.
   (b) Given that $\mathbf{EF} = \lambda\mathbf{FB}$, find λ. Evaluate the ratio $AD:FD$.
   (c) If G is a point on AE where $AG:AE$ is 1:3, find FG and hence show that C, F and G are collinear.

7. The position vectors of three points A, B and C, relative to an origin O, are $\mathbf{a}$, $\mathbf{b}$ and $2\mathbf{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC such that $CQ = 4\,QB$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vectors of P and Q and hence show that OPQ is a straight line. Given that the position vector of a point R is $\frac{5}{3}\mathbf{a}$. Show that PR is parallel to BC.

8. The position vectors of A, B and C relative to an origin O are $3\mathbf{i} + 6\mathbf{j}$, $p\mathbf{i} + 4\mathbf{j}$ and $\mathbf{i} + 5\mathbf{j}$ respectively. Given that A, B and C are collinear, find the value of p and the ratio $AB:AC$. Find the position vector of D if $\mathbf{CD}$ is $5\mathbf{i} - 12\mathbf{j}$.

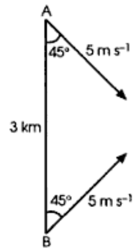
9. The three points O, P and Q are such that $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} q \\ 2q \end{pmatrix}$. Given that $\overrightarrow{PQ}$ is a unit vector, calculate the possible values of q. (C)

10. In the figure $\overrightarrow{DC} = k\mathbf{p}$ where k is a scalar.



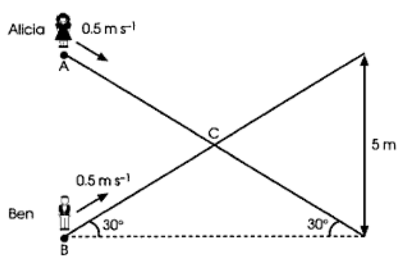
(i) Express $\mathbf{p}$ in terms of k, $\mathbf{q}$ and $\mathbf{r}$.
(ii) By expressing $\overrightarrow{FE}$ in terms of k and $\mathbf{p}$, show that FE is parallel to DC.
(iii) If $FE = 4\ AB$, find the value of k.
(iv) FA and EB are produced to meet at X. Using your value of k, express $\overrightarrow{XA}$ in terms of $\mathbf{q}$. (C)

11. At a instant, two ferries A and B are 3 km apart and are moving with constant speed of 5 m s^{-1} and in directions as shown.



(a) Find the velocity of A relative to B.
(b) Calculate the time taken for B to be due north of A, to the nearest second.

12. Escalator A is moving downwords and escalator B is moving upwards as shown below. They cross at point C and subtend an angle of 30° with the ground. They both move at a speed of 0.5 m s^{-1}. Alicia and Ben step onto escalator A and escalator B respectively, at the same time.
(a) when Alicia and Ben are both stationary relative to the respective escalators they are on,
   (i) What is Alicia's velocity relative to Ben?
   (ii) How long will they take to cross each other, i.e. to move from points A and B to point C?
(b) If Alicia climbs down the escalator at a speed of 0.3 m s^{-1} relative to escalator A, what is Alicia's velocity relative to Ben?



13. (a) A plane, with a speed of 320 km h^{-1} in still air, flies directly from London to Brussels for a distance of 320 km. The bearing of Brussels from London is 110° and there is a wind of 120 km h^{-1} blowing from the west. Find

(i) the course set by the pilot,

(ii) the time, in minutes, for the flight.

(b) At 0800 hours, a coastguard station receives a distress signal from a tanker, which is at a distance of 16 km on a bearing of 090°. The tanker is travelling at 12 km h^{-1} on a bearing of 315°. A lifeboat is immediately launched from the coastguard station to intercept the tanker. This lifeboat travels at constant speed in a straight line and intercepts the tanker at 0820 hours. Calculate the speed of the lifeboat. (C)

Chapter 14

Matrices

Curriculum Objectives:

- Display information in the form of a matrix of any order and interpret the data in a given matrix
- Solve problems involving the calculation of the sum and product (where appropriate) of two matrices and interpret the results
- Calculate the product of a scalar quantity and a matrix
- Use the algebra of 2 by 2 matrices (including the zero and identity matrix)
- Calculate the determinant and inverse of a non-singular matrix and solve simultaneous linear equations.

1. Matrix and Order

Data can be arranged as a rectangular array of numbers and represented as a **matrix**. Each number in the matrix is called an **element**.

In a matrix $\begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$, the elements are 2, 0, 1 and 5.

When a matrix has m rows and n columns, its **order** is $m \times n$.

Matrix	Number of rows, m	Number of columns, n	Order of matrix, $m \times n$
$\begin{array}{l} (1\ \ 2) \leftarrow \\ \ \uparrow\ \ \uparrow \end{array}$	1	2	1×2
$\begin{array}{l} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \\ \ \ \uparrow \end{array}$	2	1	2×1
$\begin{array}{l} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \\ \ \ \uparrow\ \ \ \uparrow \end{array}$	2	2	2×2
$\begin{array}{l} \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 5 \end{pmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \\ \ \ \uparrow\ \ \ \uparrow\ \ \ \uparrow \end{array}$	2	3	2×3

Matrix	Number of rows, m	Number of columns, n	Order of matrix, $m \times n$
$\begin{array}{cc} \begin{pmatrix} 3 & 0 \\ 2 & 5 \\ 1 & 6 \end{pmatrix} & \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \end{array} & \end{array}$	3	2	3×2
$\begin{array}{cc} \begin{pmatrix} 3 \\ 2 \\ 4 \\ 6 \end{pmatrix} & \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \end{array}$	4	1	4×1

Table 14.1

Example 1

In paper A of a Mathematics test, Mariam gets 10 correct answers, Kumar gets 9 correct answers and Cailing gets 8 correct answers. This data can be represented in a table or as a matrix.

Name	Number of correct answers
Mariam Kumar Cailing	10 9 8

or $\begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix}$

Example 2

A greengrocer has on his fruit stand: 20 large and 15 small Red and Delicious apples; 25 large and 28 small Granny Smith apples. This information can also be represented in a table or as a matrix.

	Large	Small
Red and Delicious	20	15
Granny Smith	25	28

or $\begin{pmatrix} 20 & 15 \\ 25 & 28 \end{pmatrix}$

2. Addition and Subtraction of Matrices

Matrices can be added or subtracted only when they have the same order. The resulting matrix will also have the same order.

When matrices are added or subtracted, the corresponding elements are added or subtracted.

Example 3

The number of correct answers each student gets for papers A and B of a Mathematics test is shown in the following tables.

Paper A

Name	Number of correct answers
Mariam	10
Kumar	9
Cailing	8

Paper B

Name	Number of correct answers
Mariam	12
Kumar	15
Cailing	13

The tables can be represented as two 3×1 matrices:

$$\mathbf{A} = \begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix}$$

When the teacher needs to find the total number of correct answers each student gets for the test, she simply adds the 2 matrices, i.e. **A** + **B**.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix} + \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix} = \begin{pmatrix} 10+12 \\ 9+15 \\ 8+13 \end{pmatrix} = \begin{pmatrix} 22 \\ 24 \\ 21 \end{pmatrix}$$

The total number of correct answers for Mariam is 22, for Kumar, it is 24 and for Cailing, it is 21.

Note: Matrix addition is commutative, i.e. A + B = B + A.

Example 4

The greengrocer in Example 2 finds that 5 large and 3 small Red and Delicious apples, and 1 large and 2 small Granny Smith apples are badly bruised. This information can be represented as:

	Bruised Apples	
	Large	Small
Red and Delicious	5	3
Granny Smith	7	2

or

$$\begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}$$

If he discards the bruised apples, then the remaining apples will be:

	Large	Small
Red and Delicious	20 – 5 = 15	15 – 3 = 12
Granny Smith	25 – 7 = 18	28 – 2 = 26

Alternatively, the matrices can be subtracted:
Total apples,

$$\mathbf{C} = \begin{pmatrix} 20 & 15 \\ 25 & 28 \end{pmatrix}$$

Bruised apples,

$$\mathbf{D} = \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}$$

Remaining apples

$$\begin{aligned} \mathbf{E} &= \mathbf{C} - \mathbf{D} \\ &= \begin{pmatrix} 20 & 15 \\ 25 & 28 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 20-5 & 15-3 \\ 25-7 & 28-2 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 12 \\ 18 & 26 \end{pmatrix} \end{aligned}$$

3. Multiplication of a Matrix by a Scalar Quantity

To multiply a matrix by a scalar quantity, i.e. a number, each element of the matrix is multiplied by the number.

Example 5

The Mathematics teacher in Example 1 needs to calculate the marks each student scores for paper A. Each question is allocated 3 marks.

Solution $\mathbf{A} = \begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix}$

$$\mathbf{A} \times 3 = \begin{pmatrix} 10\times 3 \\ 9\times 3 \\ 8\times 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 27 \\ 24 \end{pmatrix}$$

For Paper A, Mariam gets 30 marks; Kumar gets 27 marks and Cailing gets 24 marks.

Try Yourself:
If each question in Paper B carries 5 marks, calculate the scores for each student for Paper B. Refer to the table in Example 3 for Paper B.

4. Product of two Matrices

The product of two matrices can be found if the matrices are compatible, i.e. the number of columns in the first matrix is equal to the number of rows in the second matrix.

Pair of matrices	No. of columns in 1st matrix	No. of rows in 2nd matrix	Compatibility
$\begin{array}{c}\begin{pmatrix}1\\2\end{pmatrix}\\ \uparrow\end{array}(5\quad 6)\leftarrow$	1	1	Yes
$\begin{array}{c}\begin{pmatrix}1&2\\3&4\end{pmatrix}\\ \uparrow\quad\uparrow\end{array}(5\quad 6)\leftarrow$	2	1	No
$\begin{array}{c}(5\quad 6)\\ \uparrow\quad\uparrow\end{array}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{array}{c}\leftarrow\\ \leftarrow\end{array}$	2	2	Yes
$\begin{array}{c}\begin{pmatrix}1&2\\3&4\end{pmatrix}\\ \uparrow\quad\uparrow\end{array}\begin{pmatrix}5&6\\7&8\end{pmatrix}\begin{array}{c}\leftarrow\\ \leftarrow\end{array}$	2	2	Yes
$\begin{array}{c}\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\\ \uparrow\quad\uparrow\quad\uparrow\end{array}\begin{pmatrix}7\\8\\9\end{pmatrix}\begin{array}{c}\leftarrow\\ \leftarrow\\ \leftarrow\end{array}$	3	3	Yes
$\begin{array}{c}\begin{pmatrix}7\\8\\9\end{pmatrix}\\ \uparrow\end{array}\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{array}{c}\leftarrow\\ \leftarrow\end{array}$	1	2	No

Table 14.2

From table 14.2, the 2nd pair of matrices, $\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $(5\quad 6)$, are not compatible and cannot be multiplied. The 3rd pair of matrices, $(5\quad 6)$ and $\begin{pmatrix}1&2\\3&4\end{pmatrix}$ are compatible and their product can be found. This shows that the position of matrices is important and that matrix multiplication is non-commutative, i.e.

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}(5\quad 6)\neq(5\quad 6)\begin{pmatrix}1&2\\3&4\end{pmatrix}.$$

When two matrices **A** and **B** are compatible, they can be multiplied as follows:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$$\begin{array}{llll}
 & \mathbf{A} & \mathbf{B} & \mathbf{AB} \\
\text{1st step} & \begin{pmatrix} a & b \\ & \end{pmatrix} & \begin{pmatrix} w & \\ y & \end{pmatrix} & \begin{pmatrix} aw+by & \\ & \end{pmatrix} \\
\text{2nd step} & \begin{pmatrix} a & b \\ & \end{pmatrix} & \begin{pmatrix} & x \\ & z \end{pmatrix} & \begin{pmatrix} & ax+bz \\ & \end{pmatrix} \\
\text{3rd step} & \begin{pmatrix} & \\ c & d \end{pmatrix} & \begin{pmatrix} w & \\ y & \end{pmatrix} & \begin{pmatrix} & \\ cw+dy & \end{pmatrix} \\
\text{4th step} & \begin{pmatrix} & \\ c & d \end{pmatrix} & \begin{pmatrix} & x \\ & z \end{pmatrix} & \begin{pmatrix} & \\ & cx+dz \end{pmatrix}
\end{array}$$

$$\therefore \quad \mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$$

Example 6

Find the products of:

(i) **AB** and

(ii) **BA**, when $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Solution (i)

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times5+2\times7 & 1\times6+2\times8 \\ 3\times5+4\times7 & 3\times6+4\times8 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

(ii)

$$\mathbf{BA} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5\times1+6\times3 & 5\times2+6\times4 \\ 7\times1+8\times3 & 7\times2+8\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

Note: Example 6 shows once again that matrix multiplication is non-commutative, i.e. $\mathbf{AB} \neq \mathbf{BA}$

Example 7

To compute the total scores of each student in the Mathematics test of Example 3, the teacher may use matrix multiplication.

The teacher can combine the tables in Example 3 as a matrix: $\begin{pmatrix} 10 & 12 \\ 9 & 15 \\ 8 & 13 \end{pmatrix}$,

and represent a mark allocation of 3 marks per question in Paper A and 5 marks per question in Paper B as another matrix: $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

$$\begin{pmatrix} 10 & 12 \\ 9 & 15 \\ 8 & 13 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 10\times3+12\times5 \\ 9\times3+15\times5 \\ 8\times3+13\times5 \end{pmatrix} = \begin{pmatrix} 90 \\ 102 \\ 89 \end{pmatrix}$$

Hence the total scores of Mariam, Kumar and Cailing are 90 marks, 102 marks and 89 marks respectively.

Example 8

The greengrocer in Example 4 sells all his large apples at 30¢ each and his small apples at 20¢ each. If he sells all the remaining apples as shown in the second table, how much will he get for each type of apples?

Solution Using matrix multiplication:

Remaining apples, $\mathbf{E} = \begin{pmatrix} 15 & 12 \\ 18 & 26 \end{pmatrix}$

Price of each apple, $\mathbf{F} = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$

$$\begin{aligned} \mathbf{EF} &= \begin{pmatrix} 15 & 12 \\ 18 & 26 \end{pmatrix}\begin{pmatrix} 30 \\ 20 \end{pmatrix} \\ &= \begin{pmatrix} 15\times30+12\times20 \\ 18\times30+26\times20 \end{pmatrix} \\ &= \begin{pmatrix} 690 \\ 1060 \end{pmatrix} \end{aligned}$$

The greengrocer will get 690¢ or \$6.90 for his Red and Delicious apples and 1060¢ or \$10.60 for his Granny Smith apples.

5. The Zero Matrix and the Identity Matrix

A **zero matrix** is denoted by **O** and all its elements are zeros, e.g. a 2×2 zero matrix is written as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\mathbf{A} + \mathbf{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{A}$$

$$\mathbf{O} + \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{A}$$

i.e. $\mathbf{A} + \mathrm{O} = \mathrm{O} + \mathbf{A} = \mathbf{A}$

$$\mathbf{AO} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

$$\mathbf{OA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

i.e. $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$.

An **identity matrix** is denoted by **I**. The elements in the **leading diagonal** are ones and the other elements are zeros.

A 2×2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\mathbf{AI} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

i.e. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

In general, $\mathbf{AB} \neq \mathbf{BA}$ with the exception of the zero and identity matrices.

6. Determinant and Inverse of a Matrix

The **determinant** of a matrix **A** is denoted by $|\mathbf{A}|$, det **A** or Δ. If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|\mathbf{A}|$ is defined as $ad - bc$. When the determinant of a matrix is 0, the matrix is a **singular matrix**. The **inverse** of a matrix **A** is denoted by $\mathbf{A}^{-1}$. The product of the matrix and its inverse results in the identity matrix, i.e. $\mathbf{AA}^{-1} = \mathbf{I}$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\text{or } \frac{1}{(ad - bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note: The inverse of a matrix exists only when its determinant is not zero. The inverse of a singular matrix does not exist.

Example 9

Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 1 & 0 \end{pmatrix}$.

Solution $\quad \mathbf{A}^{-1} = \dfrac{1}{(3)(0)-(-5)(1)}\begin{pmatrix} 0 & -(-5) \\ -(1) & 3 \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} 0 & 5 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

recall:
$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note: To do a quick check, multiply $\mathbf{A}$ by $\mathbf{A}^{-1}$ and see if you get the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\mathbf{AA}^{-1} = \begin{pmatrix} 3 & -5 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3-3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7. Simultaneous Linear Equations

To solve linear equations,

$$a_1x + b_1y = c_1$$

and $\quad a_2x + b_2y = c_2$,

first write them in the form of matrices:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Next, premultiply both sides by the inverse of matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$,

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Hence values of x and y are obtained.

Example 10

Solve the simultaneous equations:

$5x + 4y = 2$
$2y = x + 8$

Solution $5x + 4y = 2$
$-x + 2y = 8$

$$\begin{pmatrix} 5 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 4 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{5(2)-4(-1)}\begin{pmatrix} 2 & -4 \\ -(-1) & 5 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 2 & -4 \\ 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & -\frac{4}{14} \\ \frac{1}{14} & \frac{5}{14} \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & -\frac{4}{14} \\ \frac{1}{14} & \frac{5}{14} \end{pmatrix}\begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{7} - \frac{32}{14} \\ \frac{1}{7} + \frac{40}{14} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{28}{14} \\ \frac{42}{14} \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$\therefore \quad x = -2$ and $y = 3$

Revision Exercises

1. Evaluate the following:

(a) $\begin{pmatrix} 5 & 2 \\ 0 & 7 \end{pmatrix} + \begin{pmatrix} 6 & 8 \\ 4 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 & 6 \\ 4 & 0 & 8 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$

(c) $2\begin{pmatrix} 1 & 6 \\ 5 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 0 & 3 \\ 1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 1 & 5 \\ 6 & -2 \end{pmatrix}$

2. Given that $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -2 & 4 \end{pmatrix} - 3\mathbf{A} = \begin{pmatrix} -6 & 11 \\ 12 & 3 \end{pmatrix}$, find the matrix **A**.

(ii) $y = \dfrac{x^2 - 2x - 3}{x^2}$

$y = 1 - \dfrac{2}{x} - \dfrac{3}{x^2}$

$y = 1 - 2x^{-1} - 3x^{-2}$

$\dfrac{dy}{dx} = (-1)(-2x^{-1-1}) + (-2)(-3x^{-2-1})$

$= 2x^{-2} + 6x^{-3}$

$= \dfrac{2}{x^2} + \dfrac{6}{x^3}$

2. Chain Differentiation or Differentiation of function of a function

To differentiate a complicated expression, we can do a chain differentiation. When y is a complicated function of x, we introduce a third variable u, such that y is a function of u and u, in turn, is a function of x.

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ **Chain rule**

Example 2

Differentiate the following functions with respect to x.

(i) $y = \sqrt[3]{3x^2 + 2x - 4}$

(ii) $y = (9x^3 + 6x^2 + 3)^5$

Solution (i) $y = (3x^2 + 2x - 4)^{\frac{1}{3}}$

Let u be $3x^2 + 2x - 4$.

$\therefore \; y = u^{\frac{1}{3}}$

Since $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{3}(3x^2 + 2x - 4)^{\frac{1}{3}-1} \dfrac{d}{dx}(3x^2 + 2x - 4)$

$= \dfrac{1}{3}(3x^2 + 2x - 4)^{-\frac{2}{3}}((2)(3x^{2-1}) + (1)(2x^{1-1}))$

$= \dfrac{1}{3(3x^2 + 2x - 4)^{\frac{2}{3}}}(6x + 2)$

$= \dfrac{2(3x + 1)}{3\sqrt[3]{(3x^2 + 2x - 4)^2}}$

$$= \left(4 - \frac{3}{x^2}\right)\left(2 - \frac{1}{x^2}\right) + \left(2x + \frac{1}{x}\right)\left(+\frac{6}{x^3}\right)$$
$$= 8 - \frac{6}{x^2} - \frac{4}{x^2} + \frac{3}{x^4} + \frac{12}{x^2} + \frac{6}{x^4}$$
$$= 8 + \frac{2}{x^2} + \frac{9}{x^4}$$

4. Quotient Rule

To differentiate the quotient of 2 functions of x,

$y = \frac{u}{v}$ where u and v are functions of x,

we may use the **quotient rule**:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Alternatively (if you find it hard to remember this formula), we can use product and chain rules to derive the formula:

$$\frac{d}{dx}(uv^{-1}) = u\left(\frac{d(v^{-1})}{dx}\right) + (v^{-1})\frac{du}{dx}$$
$$= u\left[-v^{-2}\frac{dv}{dx}\right] + (v^{-1})\frac{du}{dx}$$
$$= \frac{-u\frac{dv}{dx}}{v^2} + \frac{v\frac{du}{dx}}{v^2}$$
$$= \frac{v\frac{dv}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 4

Differentiate $y = \frac{\sqrt{2x^2 - 1}}{1 - 5x^2}$ with respect to x.

Solution Quotient method:

$$u = \sqrt{2x^2 - 1}$$
$$= (2x^2 - 1)^{\frac{1}{2}}$$
$$\frac{du}{dx} = \left(\frac{1}{2}\right)(2x^2 - 1)^{-\frac{1}{2}}(4x)$$
$$= 2x(2x^2 - 1)^{-\frac{1}{2}}$$

$$v = 1 - 5x^2$$
$$\frac{dv}{dx} = (2)(-5x)$$
$$= -10x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1-5x^2)\left[2x(2x^2-1)^{-\frac{1}{2}}\right]-(2x^2-1)^{\frac{1}{2}}(-10x)}{(1-5x^2)^2}$$

$$= \frac{(1-5x^2)\left[2x(2x^2-1)^{-\frac{1}{2}}\right]+10x(2x^2-1)^{\frac{1}{2}}}{(1-5x^2)^2}$$

$$= \frac{2x(1-5x^2)+10x(2x^2-1)}{(2x^2-1)^{\frac{1}{2}}(1-5x^2)^2}$$

$$= \frac{2x-10x^3+20x^3-10x}{(2x^2-1)^{\frac{1}{2}}(1-5x^2)^2}$$

$$= \frac{10x^3-8x}{(2x^2-1)^{\frac{1}{2}}(1-5x^2)^2}$$

5. Differentiation of Trigonometrical, Logarithmic and Exponential Functions

Trigonometrical function y (in terms of x, in rad)	Derivative, $\frac{dy}{dx}$	Note
$a\sin bx$	$ab\cos bx$	x in rad; a and b are constants
$a\cos bx$	$-ab\sin bx$	
$a\tan bx$	$ab\sec^2 bx$	
$a\cot bx$	$-ab\,\mathrm{cosec}^2 bx$	
$a\sec bx$	$ab\sec bx\tan bx$	
$a\,\mathrm{cosec}\,bx$	$-ab\,\mathrm{cosec}\,bx\cot bx$	
$a\sin(bx+c)$	$ab\cos(bx+c)$	x in rad; a, b and c are constants
$a\cos(bx+c)$	$-ab\sin(bx+c)$	
$a\tan(bx+c)$	$ab\sec^2(bx+c)$	
$\sin^n x$	$n\sin^{n-1}x\cos x$	x in rad
$\cos^n x$	$-n\cos^{n-1}x\sin x$	
$\tan^n x$	$n\tan^{n-1}x\sec^2 x$	

Logarithmic function y	Derivative, $\frac{dy}{dx}$	Note
$\log_a x$ (for $a > 0$)	$\frac{1}{x}\log_a e$	$a > 0, x > 0$
$\lg x$ (for $x > 0$) $\ln x$ (for $x > 0$)	$\frac{1}{x}\lg e$ $\frac{1}{x}$	$x > 0$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$	$f(x) > 0$

Exponential function y	Derivative, $\frac{dy}{dx}$	Note
e^x	e^x	
e^{ax}	ae^{ax}	
e^{ax+b}	ae^{ax+b}	
$e^{f(x)}$	$f'(x)\,e^{f(x)}$	

Example 5

Differentiate with respect to x

(i) $\frac{x}{\sin x}$ (ii) $\tan^2(5x+3)$ (C)

Solution (i) Let y be $\frac{x}{\sin x} = \frac{u}{v}$, where $u = x$; $v = \sin x$.

Using the quotient rule, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\sin x(1) - x(\cos x)}{\sin^2 x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$

(ii) Let y be $\tan^2(5x+3) = u^2$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 2u \times \frac{d\tan(5x+3)}{dx}$$

$$= 2u \times 5\sec^2(5x+3)$$

$$= 2\tan(5x+3)5\sec^2(5x+3)$$

$$= 10\tan(5x+3)\sec^2(5x+3)$$

Example 6

Differentiate with respect to x.

(i) $x^2 \ln(2x+1)$ (ii) $\dfrac{3x-1}{\tan x}$ (C)

Solution (i) Let y be $x^2 \ln(2x+1) = uv$, where $u = x^2$ and $v = \ln(2x+1)$

Using the product rule, $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\frac{dy}{dx} = x^2\frac{d\ln(2x+1)}{dx} + \ln(2x+1)\frac{dx^2}{dx}$$

$$\therefore \quad = x^2\frac{2}{2x+1} + 2x\ln(2x+1)$$

$$= \frac{2x^2}{2x+1} + 2x\ln(2x+1)$$

(ii) Let y be $\dfrac{3x-1}{\tan x} = \dfrac{u}{v}$, where $u = 3x-1$, $v = \tan x$.

Using the quotient rule, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\tan x\dfrac{d(3x-1)}{dx} - (3x-1)\dfrac{d\tan x}{dx}}{\tan^2 x}$$

$$= \frac{3\tan x - (3x-1)\sec^2 x}{\tan^2 x}$$

Example 7

Differentiate, $e^{2x}\tan x$ with respect to x. (C)

Solution Let y be $e^{2x}\tan x = uv$, where $u = e^{2x}$ and $v = \tan x$.

Using the product rule, $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\frac{dy}{dx} = e^{2x}\frac{d\tan x}{dx} + \tan x\frac{de^{2x}}{dx}$$

$$= e^{2x}\sec^2 x + \tan x 2e^{2x}$$

$$= e^{2x}(\sec^2 x + 2\tan x)$$

6. Equations of Tangent and Normal to a Curve

Tangent:

The gradient of tangent to a curve at any point is the gradient of the curve at that point.

Consider a point $P(x_1, y_1)$, the gradient of tangent at P is $\frac{dy}{dx} = m_1$.
The equation of the tangent is given by $y - y_1 = m_1(x - x_1)$.

Normal:
The gradient of normal at P is m_2 where $m_1 m_2 = -1$.

$\therefore\ m_2 = -\frac{1}{m_1}$

The equation of normal is $y - y_1 = m_2(x - x_1) = -\frac{1}{m_1}(x - x_1)$.

Example 8

Find the equation of the tangent and that of the normal to the curve $y = x^3 + 6x^2 + 5x + 11$ at the point (1, 2).

Solution $\quad y = x^3 + 6x^2 + 5x + 11$

$$\frac{dy}{dx} = 3x^2 + 12x + 5$$

Gradient of tangent at (1, 2) is $3(1)^2 + 12(1) + 5 = 20$

$\therefore$ Equation of the tangent is

$$\frac{y-2}{x-1} = 20$$
$$y - 2 = 20x - 20$$
$$y = 20x - 18$$

7. Small Increments and Approximation

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ where δy and δx are small increments in y and x.

If δx is very small, $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

$$\therefore\quad \delta y \approx \frac{dy}{dx} \times \delta x$$

When x changes from x_1 to $x_1 + \delta x$, then y changes from y_1 to $y_1 + \delta y$.
$\therefore$ the approximate value of y after a small increment in x of δx is

$$y_1 + \delta y = y_1 + \frac{dy}{dx} \times \delta x$$

Example 9

Given that $y = (3x + 1)^{-1}$, find the value of $\frac{dy}{dx}$ when $x = 3$. Hence find an expression, in terms of p, for the approximate change in y as x increases from 3 to $3 + p$, where p is small. (C)

Solution $y = (3x+1)^{-1}$

$\frac{dy}{dx} = (-1)(3x+1)^{-2}(3) = -\frac{3}{(3x+1)^2}$

When $x = 3$,

$\frac{dy}{dx} = -\frac{3}{(3\times 3+1)^2} = -\frac{3}{100}$

Given that $\delta x = p$ is small,

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

$$= \left(-\frac{3}{100}\right)(p) = -\frac{3p}{100}$$

8. Connected Rates of Change

Generally, differentiation is used to calculate the rate of change, for example $\frac{dy}{dx}$ is the rate of change of y with respect to x. The chain rule is often used to deduce the rate of change of one variable as compared to another.

Example 10

When the height of liquid in a tub is x metres, the volume of liquid is V m^3, where $V = 0.05((3x+2)^3 - 8)$.

(i) Find an expression for $\frac{dV}{dx}$.

(ii) The liquid enters the tub at a constant rate of 0.081 m^3 s^{-1}.
Find the rate at which the height of liquid is increasing when $V = 0.95$. (C)

Solution (i) $V = 0.05((3x+2)^3 - 8)$

$\frac{dV}{dx} = 0.05 \times 3(3x+2)^2(3) = 0.45(3x+2)^2$

(ii) $\frac{dV}{dt} = 0.081$ m^3 s^{-1}

Using chain rule,

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dx}}$$

When $V = 0.95$,

$$0.05((3x+2)^3 - 8) = 0.95$$

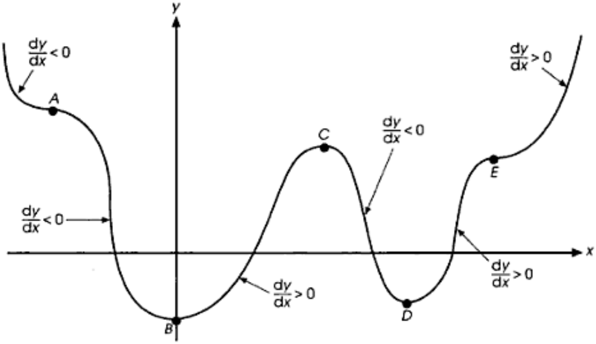
$$(3x+2)^3 = \frac{0.95}{0.05} + 8 = 27$$

$$(3x+2) = 3$$

$$\therefore x = \frac{1}{3}$$

$$\frac{dx}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dx}} = \frac{0.081}{0.45\left[3\left(\frac{1}{3}\right)+2\right]^2} = \frac{0.081}{0.45(3)^2} = 0.02 \text{ m s}^{-1}$$

9. Stationary points:



Points A, B, C, D and E are **stationary points** where $\frac{dy}{dx} = 0$ (i.e. tangents at these points are parallel to the x-axis).

Points B, C and D are **turning points**. At these points the curve turns and the gradient changes sign (i.e. change from negative to positive or vice versa).

Points B and D are local **minimum points**, being minimum in comparison with their neighbouring points.

Note that for a minimum point,

(i) $\frac{dy}{dx}$ changes from negative before the point to positive after the point.

(ii) $\frac{d^2y}{dx^2} > 0$, the rate of change of $\frac{dy}{dx}$ with respect to x is positive.

When $x = 2$,

$$\frac{d^2y}{dx^2} = -\frac{8}{(2)^5}+\frac{15}{(2)^4}-\frac{4}{(2)^3}$$

$$= -\frac{8}{32}+\frac{15}{16}-\frac{4}{8}$$

$$= \frac{-8+30-16}{32}=\frac{6}{32}=\frac{3}{16}>0$$

$\therefore$ the larger value of x corresponds to a minimum value of y.

Example 12

A rectangular block has a base which measures $2x$ cm by $3x$ cm. Given that its volume is 1800 cm^3, prove that the total surface area, A cm^2, is given by

$$A=12x^2+\frac{3000}{x}.$$

Calculate the value of x for which A has a stationary value. Find this value of A and determine whether it is a maximum or a minimum. (C)

Solution Let h be the height of the block.

Volume of block is $2x\times 3x\times h=6x^2h=1800$

$$\Rightarrow h=\frac{1800}{6x^2}=\frac{300}{x^2}$$

Total surface area, $A = 2(2x\times 3x)+2(2x\times h)+2(3x\times h)$

$$= 12x^2+4xh+6xh$$

$$= 12x^2+10xh$$

Substitute $h=\dfrac{300}{x^2}$ into A,

$$A=12x^2+10x\left(\frac{300}{x^2}\right)=12x^2+\frac{3000}{x}$$

A has a stationary value when $\dfrac{dA}{dx}=0$.

$$\frac{dA}{dx}=12(2)x+3000(-1)x^{-2}=24x-\frac{3000}{x^2}$$

When $\dfrac{dA}{dx}=0$, $24x-\dfrac{3000}{x^2}=0$

$$24x^3-3000=0$$

$$x^3=125$$

$$x=5$$

When $x = 5$, $A = 12x^2+\dfrac{3000}{x}$

$$= 12(5)^2+\frac{3000}{5}=900\text{ cm}^2$$

$$\frac{d^2A}{dx^2} = 24-(-2)\frac{3000}{x^3}=24+\frac{6000}{x^3}$$

When A is stationary, $\frac{dA}{d\theta}=0$.

$\therefore$ $16\cos 2\theta - 16\sin\theta = 0$

$\therefore$ $\cos 2\theta = \sin\theta$

$1 - 2\sin^2\theta = \sin\theta$ (since $\cos 2\theta = 1 - 2\sin^2\theta$)

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ (rejected as θ is an acute triangle)

$\therefore$ $\theta = 30°$

Revision Exercises

1. Differentiate with respect to x:
   (i) $x^3 - 5x^2 + 3x - 7$
   (ii) $5x^3 - \frac{5}{x^3}$
   (iii) $5x + 2\sqrt{x}$

2. Differentiate with respect to x:
   (i) $\sqrt{3x^2 + 2x + 1}$
   (ii) $(8 - 3x)^5$
   (iii) $(x^2 - 1)(7x^3 + 5x^2)$
   (iv) $\frac{1-2x}{3+x^2}$

3. Differentiate with respect to x:
   (i) $\cos^4 3x$
   (ii) $\sqrt{2\cos^2 x + 3}$
   (iii) $3x\sin 2x$
   (iv) $\frac{x^2}{3\tan x}$

4. Differentiate with respect to x:
   (i) $(2x+1)^4 \ln x$
   (ii) $\ln(x^2 + \sin x)$
   (iii) $e^{\cos 2x}$
   (iv) $\frac{\ln(x+2)}{x^2}$

5. Given that the curve $y = x^3 + ax^2 + bx - 3$ has a gradient of -6 at the point $(2, -11)$, find the value of a and of b.

6. Find the value of a and of b for which $\frac{d}{dx}\left(\frac{\cos x}{3+\sin x}\right)=\frac{a+b\sin x}{(3+\sin x)^2}$.

7. Find the equation of normal to the curve $y=2x^3-2x^2+5$ at the point (2, 9).

8. The equation of a curve is $y=\ln(x^2+2x)$ where $x>0$. Find the x-coordinates of the point on the curve at which the tangent to the curve is parallel to the line $5y=12x$. (C)

9. The volume, V, of a container of height, h, is given by $V=\frac{2}{3}h^3$. Find the approximate increase in volume when the height increases from 3 cm to 3.05 cm.

10. Given that $y=x^{\frac{1}{3}}$, use calculus to find an approximate value for $\frac{1}{\sqrt[3]{0.99}}$.

11. Given that $y=5-\frac{3}{x}$ and that the value of y increases from 4 by a small amount $\frac{p}{25}$, use calculus to determine, in terms of p,
    (i) the approximate change in x,
    (ii) the corresponding percentage change in x. (C)

12. Find the rate of increase of volume of a cube when the length of one side is 5 cm and the area of a face is increasing at the rate of 0.1 $cm^2\ s^{-1}$.

13. When a cylindrical vase of radius 4 cm contains water to a depth of x cm, the volume of the water in the vase is V cm^3 where $V=16\pi x$. Water is poured into the vase at the rate of 10 $cm^3\ s^{-1}$. Find the rate at which the water level is rising.

14. Air is pumped into a spherical balloon such that the latter is inflated at a steady rate of 10 $cm^3\ s^{-1}$. Find the rate of increase of its surface area when its radius is 5 cm.

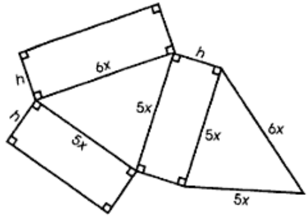
15. Given that $y=e^{3x}\cos x$, find the value of x between 0 and π for which y is stationary.

16. A curve has the equation $y=4-x^2-\frac{16}{x}$. Find the coordinates of the stationary point and determine its nature.

17. Determine the coordinates of the stationary point of the curve $y=x^2\ln x$.

18. A man wants to construct an open cylindrical fish tank for 50 fish. Given that the minimum territorial space required by per fish is 1000 cm^3, find the value of the radius, r and the height, h of this tank so that the total surface area of glass required is a minimum. Give your answer correct to 3 significant figures.

19.



A piece of cardboard is cut into the shape shown above. The cardboard is then folded along the dotted lines to form a prism of depth h and its cross-section is an isosceles triangle. Given that the volume of the prism is 1200 cm^3,

(i) show that $h = \dfrac{100}{x^2}$,

(ii) obtain an expression for S, the total surface area, in terms of x,

(iii) Find the value of x for which S has a stationary value. Find this value of S and determine whether it is a maximum or minimum.

Chapter 16

Integration

Curriculum Objectives:

- Understand integration as the reverse process of differentiation
- Integrate sums of terms in powers of x excluding $1/x$
- Integrate functions of the form $(ax + b)^n$ (excluding $n = -1$), e^{ax+b}, $\sin(ax + b)$, $\cos(ax + b)$
- Evaluate definite integrals and apply integration to the evaluation of plane areas.

Integration is the reverse process of differentiation.

1. Indefinite Integrals

When we integrate f(x) with respect to x, i.e. $\int f(x)dx$,

$\int \mathbf{f}(\boldsymbol{x})\mathbf{d}\boldsymbol{x} = \mathbf{F}(\boldsymbol{x}) + \boldsymbol{c}$ is called an indefinite integral.

Note: c is a constant.

f(x)	F(x) + c	Note
ax^n	$\frac{ax^{n+1}}{n+1} + c$	$n \neq 1$
a	$ax + c$	
$(ax + b)^n$	$\frac{(ax+b)^{n+1}}{(n+1)(a)} + c$	$n \neq -1$, $a \neq 0$; a and b are constants
$a\cos bx$ $a\sin bx$ $a\sec^2 bx$	$\frac{a}{b}\sin bx + c$ $-\frac{a}{b}\cos bx + c$ $\frac{a}{b}\tan bx + c$	x in radians; a, b and c are constants
$a\mathrm{cosec}^2 bx$ $a\sec bx \tan bx$ $a\mathrm{cosec} bx \cot bx$	$-\frac{a}{b}\cot bx + c$ $\frac{a}{b}\sec bx + c$ $-\frac{a}{b}\mathrm{cosec} bx + c$	

$a\cos(bx+c')$	$\frac{a}{b}\sin(bx+c')+c$	x in radians; a, b, c and c' are constants
$a\sin(bx+c')$	$-\frac{a}{b}\cos(bx+c')+c$	
$a\sec^2(bx+c')$	$\frac{a}{b}\tan(bx+c')+c$	
$\frac{1}{x}$	$\ln x+c$	For $x>0$
$\frac{f'(x)}{f(x)}$	$\ln f(x)+c$	For $f(x)>0$
e^x	e^x+c	
e^{ax}	$\frac{1}{a}e^{ax}+c$	
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$	
$f'(x)e^{f(x)}$	$e^{f(x)}+c$	

Note: Compare this table with the differentiation table.

Sum of integrals:

$$\int f(x)dx \pm \int g(x)dx = \int (f(x) \pm g(x))dx$$

Example 1

Integrate the following with respect to x:

(i) 4 (ii) $3x^2$ (iii) $2x(x-1)^2$

(iv) $\frac{x^4-x}{x^3}$ (v) $\sqrt{x^3}+\frac{2}{\sqrt{x}}$ (vi) $(3x+5)^5$

(vii) $\left(\frac{3}{2x-1}\right)^4$

Solution (i) $\int 4dx = 4x+c$

(ii) $\int 3x^2dx = \frac{3x^3}{3}+c$
$= x^3 + c$

(iii) $\int 2x(x-1)^2dx = \int 2x(x^2-2x+1)dx$
$= \int (2x^3-4x^2+2x)dx$

(iv) $\int \frac{5\sec^2 2x}{2\tan 2x+4}dx = \frac{5}{4}\int \frac{4\sec^2 2x}{2\tan 2x+4}dx$

$= \frac{5}{4}\ln(2\tan 2x+4)+c$

Example 4

Integrate with respect to x

(i) $6e^{4x}$

(ii) $\frac{1}{e^{2x-1}}$

Solution (i) $\int 6e^{4x}dx = \frac{6}{4}e^{4x}+c$

$= \frac{3}{2}e^{4x}+c$

(ii) $\int \frac{1}{e^{2x-1}}dx = \frac{1}{2}\ln(e^{2x-1})+c$

2. Definite Integrals

When we integrate f(x) with respect to x from $x = a$ to $x = b$, i.e. $\int_a^b f(x)dx$, we get a definite integral: $\int_a^b f(x)dx = F(b) - F(a)$.

Example 5

Evaluate

(i) $\int_0^4 \frac{5x+3}{\sqrt{x}}dx$ (ii) $\int_0^{\frac{\pi}{2}} \left(3x+\sin\frac{x}{2}\right)dx$

(iii) $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \tan^2\frac{x}{2}dx$ (iv) $\int_0^1 \frac{x}{3-x^2}dx$

(v) $\int_0^1 3e^{-2x}dx$

Solution (i) $\int_0^4 \frac{5x+3}{\sqrt{x}}dx = \int_0^4\left(5\sqrt{x}+\frac{3}{\sqrt{x}}\right)dx$

$= \left[\frac{2}{3}(5x^{\frac{3}{2}})+(2)3x^{\frac{1}{2}}\right]_0^4$

$= \left[\frac{10}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}\right]_0^4$

$= \frac{10}{3}(4)^{\frac{3}{2}}+6(4)^{\frac{1}{2}}-0$

$= \frac{80}{3}+12 = 38\frac{2}{3}$

(ii) $$\int_0^{\frac{\pi}{2}}\left(3x+\sin\frac{x}{2}\right)dx = \left[\frac{3}{2}x^2 - 2\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{3}{2}\left(\frac{\pi}{2}\right)^2 - 2\cos\frac{\pi}{4} - (-2\cos 0)$$
$$= 3.701 - 1.414 + 2$$
$$= 4.29$$

(iii) $$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \tan^2\frac{x}{2}dx = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(\sec^2\frac{x}{2}-1\right)dx$$
$$= \left[2\tan\frac{x}{2} - x\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$
$$= 2\tan\frac{\pi}{3} - \frac{2\pi}{3} - \left(2\tan\frac{\pi}{4} - \frac{\pi}{2}\right)$$
$$= 2(1.732) - 2 - \frac{2\pi}{3} + \frac{\pi}{2}$$
$$= 0.941$$

(iv) $$\int_0^1 \frac{x}{3-x^2}dx = -\frac{1}{2}\int_0^1 \frac{-2x}{3-x^2}dx$$
$$= -\frac{1}{2}(\ln(3-x^2))_0^1$$
$$= -\frac{1}{2}\ln 2 + \frac{1}{2}\ln 3$$
$$= 0.203$$

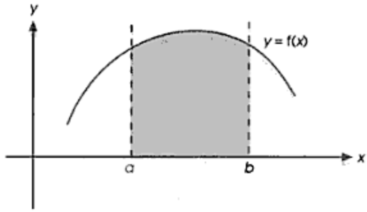
(v) $$\int_0^1 3e^{-2x}dx = \left[-\frac{3}{2}e^{-2x}\right]_0^1$$
$$= -\frac{3}{2}e^{-2} - \left(-\frac{3}{2}e^0\right)$$
$$= \frac{3}{2} - \frac{3}{2}(0.135)$$
$$= 1.30$$

3. Area under a Curve

While a derivative of a function is the gradient of the function, an integral of a function is the area bounded by the curve.

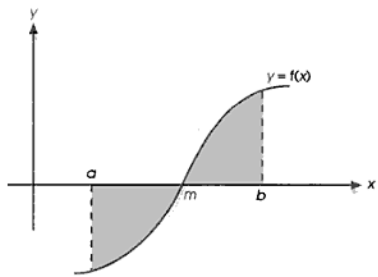
A. *Area between Curve and the x-axis*

When we integrate f(x) with respect to x from $x = a$ to $x = b$, the integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis.



In the case where the curve intersects the x-axis at a point m within the boundaries of $x = a$ and $x = b$, i.e. $a < m < b$, the algebraic area bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is

$$\int_a^b f(x)dx = \int_a^m f(x)dx + \int_m^b f(x)dx$$



Total area bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is

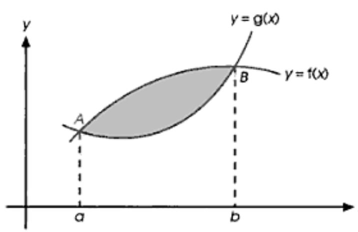
$$\left|\int_a^m f(x)dx\right| + \left|\int_m^b f(x)dx\right|$$

B. *Area Bounded by Two Curves*

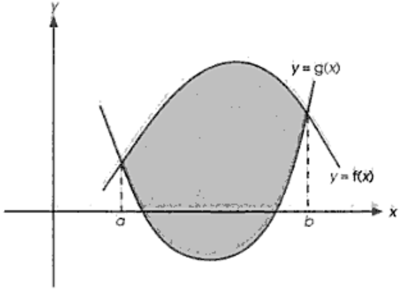
Two curves $y = f(x)$ and $y = g(x)$ intersect at the points A and B (corresponding to $x = a$ and $x = b$ respectively). The area bounded by the two curves is
(Area under $y = f(x)$ between $x = a$ and $x = b$) – (Area under $y = g(x)$ between $x = a$ and $x = b$)

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b (f(x) - g(x))dx$$

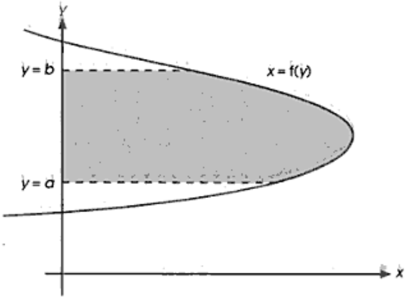


Note: the integral $\int_a^b (f(x) - g(x))dx$ gives the total area of the region between the curves.



C. *Area between a Curve and the y-axis*

The area enclosed by the curve $x = f(y)$, the lines $y = a$ and $y = b$ and the y-axis is is given by $\int_a^b x dy = \int_a^b f(y)dy$.



Example 6

(a) Evaluate, giving your answers correct to two decimal places.

(i) $\int_0^1 \frac{8}{3x+2}dx$

(ii) $\int_1^2 e^{-\frac{1}{2}x}dx$

$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \pm\frac{1}{\sqrt{2}} \qquad \text{(according to the diagram, } x \text{ is positive)}$$

$$x = \frac{\pi}{4}$$

(bii) Area of the shaded region $= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$

$$= [\sin x - (-\cos x)]_0^{\frac{\pi}{4}}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

$$= 0.414 \text{ unit}^2$$

Revision Exercises

1. Find

(a) $\int \sqrt{x}\left(x^2 + \frac{1}{x}\right)dx$ (b) $\int \frac{3-x}{x^2}dx$

(c) $\int \left(\frac{1}{x^3} - \frac{1}{x^4}\right)dx$ (d) $\int \frac{(1-\sqrt{x})(1+\sqrt{x})}{x}dx$

2. Evaluate

(a) $\int_1^2 \left(2x^4 - \frac{6}{x^2}\right)dx$ (b) $\int_4^9 \frac{2x^2 - 3\sqrt{x}}{x^2}dx$

(c) $\int_0^4 x(x - 3\sqrt{x})^2 dx$

3. Given that $\int_2^5 f(x)dx = 12$ and $\int_2^5 g(x)dx = 5$, state which of the following integrals cannot be evaluated and hence evaluate the others.

(i) $\int_2^5 3f(x)dx$ (ii) $\int_2^5 \{f(x) + g(x)\}dx$

(iii) $\int_2^5 f(x).g(x)dx$ (iv) $\int_5^2 g(x)dx$

(v) $\int_2^5 \{f(x) + 1\}dx$ (vi) $\int_2^{10} f(x)dx$

4. Evaluate

(i) $\int_0^{\frac{\pi}{6}} \{\sin 3x - 2\cos x\}dx$ (ii) $\int_0^{\frac{\pi}{6}} 4\sin 2x dx$

(iii) $\int_0^{\frac{\pi}{9}} \cos\left(3x - \frac{\pi}{6}\right)dx$ (iv) $\int_0^{\frac{\pi}{8}} \frac{\sec^2 2x}{2}dx$

5. A curve is such that $\frac{dy}{dx} = 4x - 3$ and it passes through the point $P(1, -4)$. Find

(i) the equation of the normal at P,

(ii) the equation of the curve.

6. The curve for which $\frac{dy}{dx} = 3x^2 + ax + b$ where a and b are constants, has stationary points at (1, 0) and (–3, 32). Find

(i) the value of a and of b,

(ii) the equation of the curve. (C)

7. Given that $y = 2e^x(\sin x - \cos x)$, show that $\frac{dy}{dx} = 4e^x \sin x$. Hence evaluate $\int_0^{\pi} e^x \sin x \, dx$.

8. Given that $y = 2xe^{-x}$,

(i) show that y has a stationary value when $x = 1$,

(ii) complete the following table.

x	0	0.5	1	2
y				

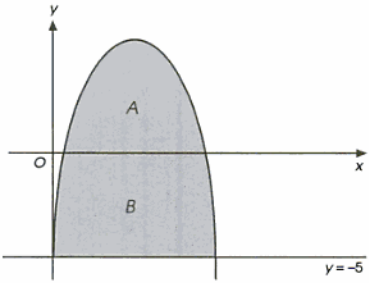
Using graph paper, draw the graph $y = 2xe^{-x}$ for $0 \le x \le 2$. By drawing a suitable line, find the solutions of the equation

$x + 1 = 10xe^{-x}$. (C)

9. Given that a curve has the equation $y = 2e^x(x + 3)$, find $\frac{dy}{dx}$ and hence find the coordinates of the stationary point on the curve. State the coordinates of the points at which the curve crosses the x and y-axes. Sketch the curve for $-5 \le x \le 1$.

10. The figure shows part of the curve $y = 6x - x^2 - 5$. Calculate the shaded area marked

(i) A, (ii) B.

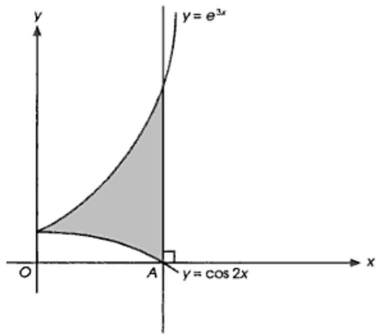


(C)

11. Find the area enclosed by the curve $y = e^{2x-1}$, the x and y-axes and the line $x = 1$.

12. (a) Evaluate

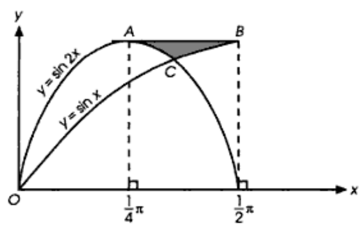
(i) $\int_3^4 (5-x)^5 dx$ (ii) $\int_{-1}^0 \frac{4}{2x+3} dx$



(b) The diagram shows part of the graphs of $y = e^{3x}$ and $y = \cos 2x$. Find

(i) the x-coordinate of A,

(ii) the area of the shaded region.

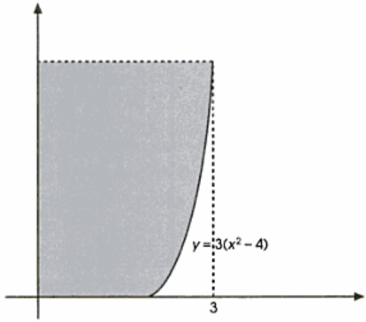


(C)

13. The diagram shows part of the graphs of the curves $y = \sin 2x$ and $y = \sin x$.
A is a maximum point of the curve $y = \sin 2x$.
B is a maximum point of the curve $y = \sin x$.
C is the point of intersection of the curves shown.
Find

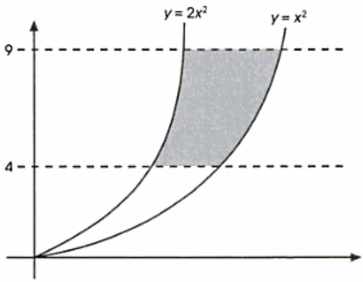
(i) the x-coordinate of C,

(ii) the shaded area.



(C)

14. The diagram shows part of the curve $y = 3(x^2 - 4)$. Calculate the area of the shaded region.



15. The diagram shows part of the graphs of the curves $y = 2x^2$ and $y = x^2$. Find the area enclosed by the curves and the lines $y = 4$ and $y = 9$.

Chapter 17

Kinematics

Curriculum Objectives:

- Apply differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration and the use of $x - t$ and $v - t$ graphs.

Kinematics is the study of the motion of a body without considering the cause of its motion. To describe the motion of a body, we need to know its distance from a point of reference, its speed, how its speed changes with time and the diection of its motion. Vectors like displacement, velocity and acceleration are terms used to describe its motion.

Distance and its direction are indicated by **displacement, *s*.**

Speed and its direction are indicated by **velocity, *v*.**

Rate of change of speed with respect to **time and its direction** are indicted by **acceleration, *a*.**

In this chapter, we are concerned with the motion of a body along a straight line.

1. Velocity

When a body travels with a constant velocity, its average velocity,

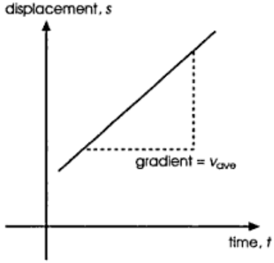
$$v_{ave} = \frac{\text{change in displacement}}{\text{change in time}}$$

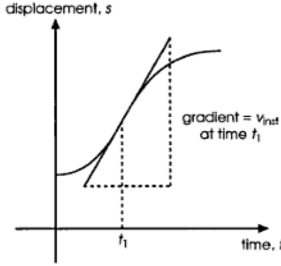
$$= \frac{\Delta s}{\Delta t}$$

However if its velociity changes with time, its instantaneous velocity, v_{inst}, is the rate of change of displacement with respect to time at a given instant.

$$v_{inst} = \frac{ds}{dt}$$

Average velocity and instantaneous velocity are given by the gradients of displacement–time, $s - t$, graphs.





In general

> **velocity** is the **rate of change of displacement with** respect to **time**, i.e. $\boldsymbol{v = \frac{ds}{dt}}$.

2. Acceleration

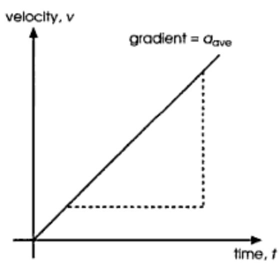
Similarly, when a body travels with constant acceleration, its average acceleration, a_{ave} is

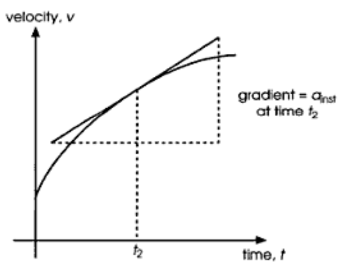
$$a_{ave} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{\Delta v}{\Delta t}.$$

When its acceleration varies with time, its instantaneous acceleration, a_{inst} is

$$a_{inst} = \frac{dv}{dt}.$$

Average acceleration and instantaneous acceleration are given by the gradients of speed – time, $v - t$, graphs:





In general,

acceleration is the **rate of change of velocity with** respect to **time**, i.e. $$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Note: When acceleration is negative ($a < 0$), it is called **deceleration** or **retardation**.

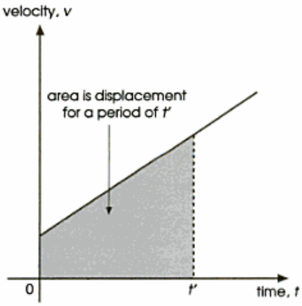
3. Displacement, Velocity and Acceleration

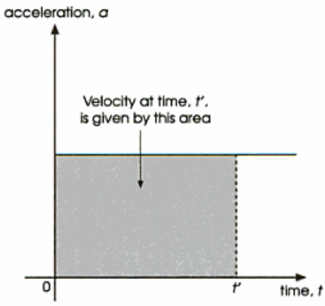
When displacement, s, velocity, v, and acceleration, a, are functions of time, we have

$$v = \frac{ds}{dt} \quad \Rightarrow \quad s = \int v \, dt$$

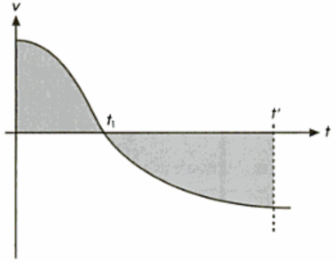
$$a = \frac{dv}{dt} \quad \Rightarrow \quad v = \int a \, dt$$

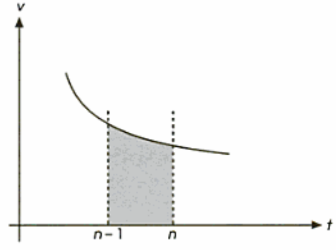
Alternatively, displacement is given by area under the velocity–time ($v - t$) graph and velocity is given by area under the acceleration–time ($a - t$) graph:





4. Distance and Displacement



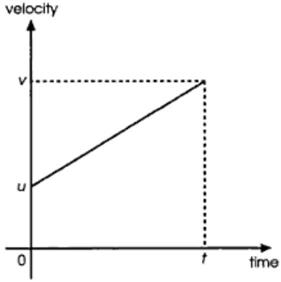


The total distance travelled by a particle from $t=0$ to $t=t'$ is $\left|\int_0^{t_1} v\,dt\right| + \left|\int_{t_1}^{t'} v\,dt\right|$.

The total displacement travelled by a particle from $t=0$ to $t=t'$ is $\int_0^{t'} v\,dt$.

The displacement of a particle in the n^{th} second is $\int_{n-1}^{n} v\,dt$.

5. Equations of Motion with Constant Acceleration



a = gradient of graph

$$= \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \frac{v-u}{t}$$

$\Rightarrow$ $\boldsymbol{v = u + at}$ .... (1)

s = area under $v-t$ graph

$\boldsymbol{s = \frac{1}{2}(u+v)t}$ .... (2)

substitute (1) into (2).

$$s = \frac{1}{2}(u+(u+at))t$$

$\boldsymbol{s = ut + \frac{1}{2}at^2}$ .... (3)

To eliminate t, make t the subject of (1).

$\boldsymbol{t = \frac{v-u}{a}}$ .... (4)

Substitute (4) into (2).

$$s = \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right)$$

$2as = v^2 - u^2$

$\boldsymbol{v^2 = u^2 + 2as}$ .... (5)

In summary, the following equations apply when a = constant.

Equations	Variables
$v = u + at$	u, v, a, t
$s = \frac{1}{2}(u + v)t$	s, u, v, t
$s = ut + \frac{1}{2}at^2$	s, u, a, t
$v^2 = u^2 + 2as$	s, u, v, a

Tip: Sometimes it may be confusing to know which equation to apply. By looking at the variables involved in the equation, you would be able to guess which equation to use. Often, there is more than one method of solving kinematics problems.

Example 1

A particle starts from point O and moves in a straight line so that its displacement, s cm, from O, t seconds after leaving O, is given by $s = t(t-6)^2$. Obtain an expression for the velocity of the particle in terms of t. Hence determine the value of t when the particle first comes to instantaneous rest and find the acceleration at this instant. The particle is next at O when $t = T$. Find

(i) the value of T,

(ii) the distance travelled from $t = 0$ to $t = T$. (C)

Solution

$$\begin{aligned} s &= t(t-6)^2 \\ &= t(t^2 - 12t + 36) \\ &= t^3 - 12t^2 + 36t \\ v &= \frac{ds}{dt} \\ &= 3t^2 - 24t + 36 \end{aligned}$$

At instantaneous rest, $v = 0$,

$$\begin{aligned} 3t^2 - 24t + 36 &= 0 \\ 3(t^2 - 8t + 12) &= 0 \\ 3(t-2)(t-6) &= 0 \\ \Rightarrow \quad t = 2 \quad \text{or} \quad t &= 6 \end{aligned}$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d}{dt}(3t^2 - 24t + 36) \\ &= 6t - 24 \end{aligned}$$

When the particle first comes to instantaneous rest, $t = 2$.

$a = 6(2) - 24$
$= -12 \text{ m s}^{-2}$

(i) Given that the particle is next at O when $t = T$,

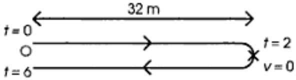
$\therefore \ s = 0$
$s = t(t - 6)^2 = 0$
$t = 0$ or $t = 6$
$\therefore \ T = 6$ s

(Note when $t = 0$, the particle starts from point O.)

(ii) To find the distance travelled from $t = 0$ to $t = 6$, consider $t = 2$ s. At this point, $v = 0$ and $a = -12 \text{ m s}^{-2} < 0$. This implies that the particle's displacement from point O is maximum when $t = 2$ s.

$s = t(t - 6)^2$
$= 2(2 - 6)^2$
$= 32$ m

Distance travelled from $t = 0$ to $t = 2$ (when the particle first comes to a rest) is 32 m. It follows that the distance travelled (from $t = 2$ to $t = T = 6$) when the particle returns to point O is also 32 m.



$\therefore$ Total distance travelled from $t = 0$ to $t = T = 6$ is $32 \times 2 = 64$ m.

Example 2

A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its velocity, $v \text{ m s}^{-1}$, is given by $v = 15\sin\frac{1}{3}t$.

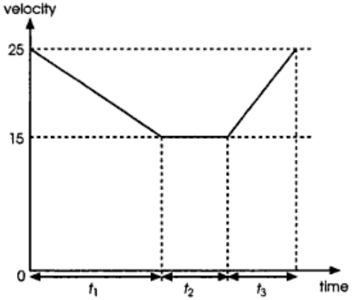
Find

(i) the time at which the particle first has a speed of 10 m s^{-1},
(ii) the acceleration of the particle when $t = 0$,
(iii) an expression for the displacement of the particle from O in terms of t.

(C)

(a) the time during which the train is accelerating,
(b) the speed of the train 180 seconds after the start of this stage,
(c) the speed of the train 660 seconds after the start of this stage.(C)

Solution



(a) Total distance travelled = 12 000 m

Total time = 720 s

Let t_1, t_2 and t_3 be the duration of the 3 parts of the journey as shown in the graph.

$$t_1 + t_2 + t_3 = 720 \text{ s}$$

Let a_1 be the deceleration and a_2 be the acceleration.

Given that $|a_2| = |2a_1|$

$$\Rightarrow t_1 = 2t_3$$

$$\therefore (2t_3) + t_2 + t_3 = 720$$

$$\Rightarrow t_2 + 3t_3 = 720$$

Total distance travelled

$$\begin{aligned}
&= \text{area under } v - t \text{ graph} \\
&= \frac{1}{2}(15 + 25)t_1 + 15t_2 + \frac{1}{2}(15 + 25)t_3 \\
&= \frac{1}{2}(40)t_1 + 15t_2 + \frac{1}{2}(40)t_3 \\
&= 20(2t_3) + 15t_2 + 20t_3 \\
&= 15t_2 + 60t_3 \\
&= 15(720 - 3t_3) + 60t_3 \\
&= 10\,800 - 45t_3 + 60t_3 \\
&= 10\,800 + 15t_3 \\
&= 12\,000
\end{aligned}$$

$$\therefore \quad t_3 = \frac{12\,000 - 10\,800}{15}$$

$$= 80 \text{ s}$$

Time during which train is accelerating is 80 s.

Velocities of Q:

at A, $v_{Q,A} = u$

at B, $v_{Q,B} = 22 \text{ m s}^{-1}$

Given that car Q overtakes car P at B, i.e. car P and Q reaches point B at the same time, t,

$$20t = \frac{1}{2}(u + 22)t$$

$$\therefore \quad u = 40 - 22$$

$$= 18 \text{ m s}^{-1}$$

(a) Consider the time taken by each car to travel from point B to point D:

time taken by car P is $\frac{750}{20} = 37.5$ s

Car Q takes 15 s to travel from point B to point C.

Let t' be the time taken for it to travel from point C to point D.

Given that car Q decelerates uniformly from 22 m s^{-1} at point C to 13 m s^{-1} at point D, average speed between point C and point D is $\frac{22+13}{2}$.

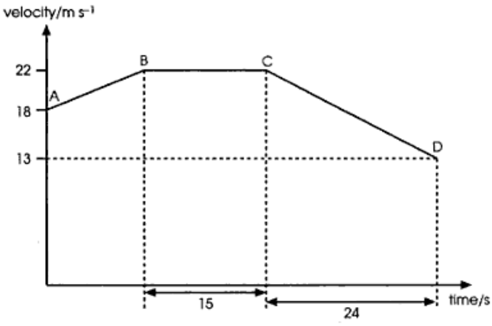
$$750 = 22(15) + \left(\frac{22+13}{2}\right)t'$$

$$\therefore \quad t' = 24 \text{ s}$$

Time taken for car Q to travel from point B to point D is $15 + 24 = 39$ s.

$\therefore$ Car P reaches D before car Q.

(b)



(c) Let E be the point (between B and D) that car Q reaches, 37.5 seconds after passing B.

$s_{B,E} = s_{B,C} + s_{C,E}$

$s_{B,C} = 22(15)$

Using $s = u't + \frac{1}{2}at^2$

$$s_{C,E} = 22(37.5 - 15) + \frac{1}{2}\left(\frac{13-22}{24}\right)(37.5 - 15)^2$$

$$= 400.1 \text{ m}$$

$\therefore\ s_{B,E} = 22(15) + 400.1 = 730.1$ m

Distance of car Q from D at the instant when car P reaches D = 750 − (730.1)

= 20 m (to nearest metre)

Alternatively, let v be the speed of car Q at the instant when car P reaches D.

Using $v = u' + at$,

$$v = 22 + \left(\frac{13-22}{24}\right)(37.5 - 15)$$

$$= 13.56 \text{ m s}^{-1}$$

Using $s = u''t'' + \frac{1}{2}at''^2$,

distance car Q is from D

$$= (13.56)\ (39 - 37.5) + \frac{1}{2}\left(\frac{13-22}{24}\right)(39 - 37.5)^2$$

= 20 m (to nearest metre)

Revision Exercises

1. A particle travels in a straight line in such a way that t seconds after passing through a fixed point O, its displacement from O is s metres. Given that $s = 5 - \dfrac{5}{(t+1)}$, find
   (i) expressions, in terms of t, for the velocity and acceleration of the particle,
   (ii) the value of t when the velocity of the particle is 1.25 m s^{-1},
   (iii) the acceleration of the particle when it is 3 m from O.

2. A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its displacement, s m, is given by

   $$s = 5 - 5e^{-t} - \frac{1}{10}t$$

   Calculate
   (i) the initial velocity of the particle,
   (ii) the value of t when the particle is instantaneously at rest,
   (iii) the acceleration of the particle at this instant. (C)

8. A motorcyclist travelling along a straight road passes a fixed point O with a speed of 20 m s^{-1} and continues at this speed for t_1 seconds. Over the next t_2 seconds he accelerates at a constant rate to a speed of 30 m s^{-1}. He then brings the motorcycle to rest in a further t_3 seconds by retarding at a constant rate. His acceleration and retardation are of equal magnitude.

(a) Sketch a velocity-time graph to illustrate the motion of the motorcyclist after passing O.

(b) Obtain an equation connecting t_2 and t_3.

(c) Given that the total distance and the total time represented by the graph are 748 m and 40 s respectively, calculate t_1, t_2 and t_3. (C)

Specimen Paper A

The topics from which the questions are set are included in brackets { }.

Paper 1 (80 marks) **(Time: 2 hours)**

Answer **all** the questions.

{T6: Simultaneous Equations}

1. Solve the simultaneous equations:

$$y = x^2 + 6x + 4$$
$$3y - x = 2$$

(4)

{T8: Straight Line Graphs}

2. The points A, B and C have coordinates $(-4, 2)$, $(2, 8)$ and $(5, -1)$ respectively. The line from C, which is perpendicular to AB, meets AB at a point D.
   (i) Find the equation of AB and of CD.
   (ii) Calculate the coordinates of D. (6)

{T3: Quadratic Functions}

3. If the equation $x^2 + (1 - k)x + 3 = 0$ has no real roots, find the range of the values of k. (4)

{T15: Differentiation}

4. Variables x and y are connected by the equation $y = 2x + \frac{4}{x^2}$. Calculate the value of $\frac{dy}{dx}$ when $x = 2$. Hence find an expression for the new value of y when x increases from 2 to $2 + p$ where p is small. (6)

{T2: Binomial Expansions}

5. Find
   (i) the coefficient of x^2 in the expansion of $\left(x - \frac{3}{x}\right)^{10}$
   (ii) the coefficient of x^3 in the expansion of $(1+2x)\left(x - \frac{3}{x}\right)^{10}$ (6)

{T17: Kinematics}

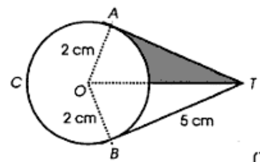
6. A particle moves in a straight line through a fixed point O. Its acceleration, a m s^{-2}, is given by $a = (2t + 5)$ m s^{-2}, where t is the time in seconds after passing through O. Given that a particle reaches a point X when $t = 1$ with a velocity 10 m s^{-1}, find
   (i) the velocity of the particle when $t = 2$,
   (ii) the distance OX. (7)

{T10: Trigonometry}

7. Given that $\tan x = p$, where x is acute, find in terms of p,
   (i) $\tan(\pi - x)$
   (ii) $\sec x$
   (iii) $\sin x$ (5)

{T9: Circular measure}

8. The diagram shows a circle, centre O, radius 2 cm, and two tangents TA and TB. $TA = TB = 5$ cm.
   Calculate
   (i) the length of the major arc ACB,
   (ii) the area of the shaded region. (7)

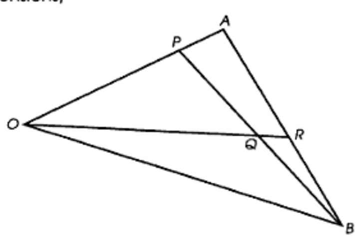


{T1: Sets}

9. Given that n(ε) = 60, n(A) = x, n(B) = 35 and n(A ∩ B) = 8, express in terms of x, n(A ∩ B) and n(A′ ∩ B′). Hence find the greatest and smallest possible values of x. (8)

{T13: Vectors in Two Dimensions}

10.



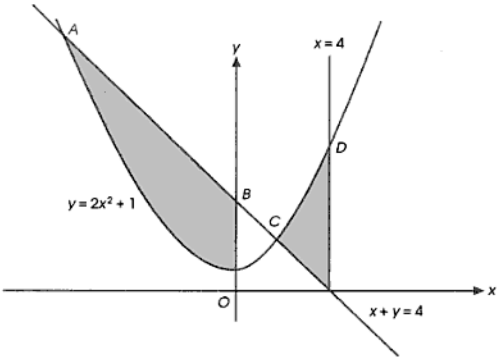
In the figure shown the position vectors of A and B with respect to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. The points P and Q are such that $\mathbf{OP} = \frac{3}{4}\,\mathbf{OA}$ and $2\mathbf{PQ} = \mathbf{PB}$. Express $\mathbf{AP}$ and $\mathbf{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Given that $\mathbf{OR} = \lambda\mathbf{OQ}$ and $\mathbf{BR} = \mu\mathbf{AB}$, express $\mathbf{BR}$ in terms of
(i) λ, $\mathbf{a}$, $\mathbf{b}$ (ii) μ, $\mathbf{a}$, $\mathbf{b}$
Hence evaluate λ and μ. (10)

{T2: Functions}

11. The function f is defined by $f: x \to \dfrac{x^2}{4}$ for $0 \le x \le 6$.

Sketch the graphs of f and f^{-1} on the same diagram. State the domain of f^{-1}. (5)

12. Answer only one of the following two alternatives.
Either:



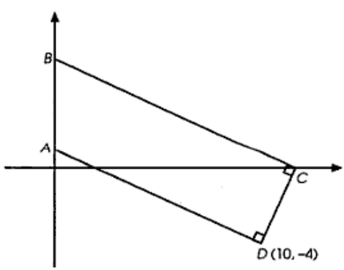
{T16: Integration}

The diagram shows part of the curve $y = 2x^2 + 1$ and the lines $x + y = 4$ and $x = 4$. Find

(i) the coordinates of the points A, B, C and D,

(ii) the area of each of the shaded regions. (12)

Or:



{T8: Straight Line Graphs}

The diagram shows a trapezium $ABCD$ in which BC is parallel to AD. The points A

and B lie on the y-axis and point C lies in the x-axis. Angles BCD and CDA are $90°$. Given that the equation of BC is $2y + x = 12$ and point D is $(10, -4)$,

(i) find the coordinates of B and of C,

(ii) find the equation of AD and of CD,

(iii) the coordinates of A,

(iv) the area of trapezium. (12)

Paper 2 (80 marks) **(Time: 2 hours)**

Answer **all** the questions.

{T5: Factors of Polynomials}

1. Find in terms of p, the remainder when $6x^3 + 5x^2 + px - 6$ is divided by $x + 2$. Hence write down the value of p for which the expression is exactly divisible by $x + 2$. (6)

{T8: Straight Line Graphs}

2 The table shows experimental values of two variables x and y.

x	1	2	3	4	5	6
y	4.1	2.8	2.2	1.7	1.6	1.3

It is known that x and y are related by the equation $y = \dfrac{a}{x+b}$, where a and b are constants.

(i) Plot y against xy and obtain a straight line graph.

(ii) Use your graph to estimate the value of a and of b.

(iii) Obtain the value of the gradient of the straight line obtained when $\dfrac{1}{y}$ is plotted against x. (10)

{T8: Trigonometry}

3. Find all the angles between 0° and 360° which satisfy the equation

(i) $3\sin x\cos x + 1 = 0$

(ii) $\tan 2y = 6\cot y$ (8)

{T7: Logarithmic and Exponential Functions}

4. Given that $y = 120(0.98)^x$, find

(i) the value of y when $x = 20$,

(ii) the value of x when $y = 100$. (4)

{T2: Functions}

5. (a) The function g is defined by g: $x \rightarrow \dfrac{2x}{x+3}$, $x \neq -3$.
Find and simplify an expression for

(i) g^2 (ii) g^{-1}

(b) The function h is defined by h: $x \to ax - 1$.

Given that $hg^2(2) = \frac{5}{19}$, calculate the value of a. (7)

{T11: Permutations and Combinations}

6. A basket has 5 red apples and 4 green apples. Find the number of ways of choosing 4 apples when
   (i) there are no restrictions;
   (ii) there must be 1 red apple only;
   (iii) there must be 2 green apples. (8)

{T14: Matrices}

7. Given that $A = \begin{pmatrix} 2 & 1 \\ -5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$ and $ABC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
   (a) Find the matrix AB.
   (b) Find the matrix C. (6)

{T16: Integration}

8. Evaluate
   (i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos 2x - 3\sin x)dx$
   (ii) $\int_0^{\pi} \cos x e^{\sin x} dx$ (8)

{T17: Kinematics}

9. A particle moves in a straight line so that at time t after leaving a fixed point O, its velocity, v m s^{-1}, is given by $v = 2 - 2e^{-\frac{1}{2}t}$.
   (i) Find the acceleration of the particle when $t = 2$ s.
   (ii) Sketch the velocity-time curve.
   (iii) Find the displacement of the particle from O when $t = 2$ s. (8)

{T13: Vectors in Two Dimensions}

10. Given that the vector $\boldsymbol{OA}$ has a magnitude of 20 and has the same direction as $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and that $\boldsymbol{OB}$ is a vector of magnitude 10 and in the direction of the vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, find the vector $\boldsymbol{AB}$. (3)

11. Answer only one of the following two alternatives.

    Either

    {T15: Differentiation}

    (a) Find the value of x between 0 and p for which the curve $y = e^x \cos 2x$ has a stationary point.

Specimen Paper B

Paper 1 (80 marks) **(Time: 2 hours)**

Answer **all** the questions.

{T8: Straight Line Graphs}

1. Given points $A\,(-1, -9)$, $B\,(6, 12)$ and $C\,(3, -2)$, find the equation of the line joining A to B. Another line passes through C and meets AB at right angle at D. Find the equation of CD and calculate the coordinates of D. (6)

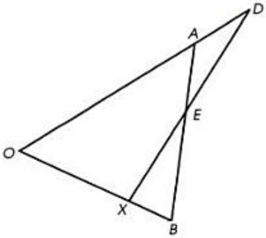
{T2: Functions}

2. Express $8 - 4x - x^2$ in the form $a - (x + b)^2$ and hence, or otherwise, find the range of the function f: $x \to 8 - 4x - x^2$ for real x. (4)

{T12: Binomial Expansions}

3. Write down the binomial expansions of $(2 + x)^4$ and $(2 - x)^4$, in ascending powers of x. Use your results to calculate the exact value of $(2.1)^4 - (1.9)^4$. (7)

{T13: Vectors in Two Dimensions}



4. The position vectors of three points, O,A and B are $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}4\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$ respectively. Given that OD = $\frac{4}{3}$ OA and AE = $\frac{1}{2}$ AB, write down the position vectors of D and E.
Given also that OB and DE intersect at X and that $\boldsymbol{OX} = p\boldsymbol{OB}$ and $\boldsymbol{XD} = q\boldsymbol{DE}$, find the position vector of X in terms of
(i) p (ii) q
Hence calculate p and q. (8)

{T10: Trigonometry}

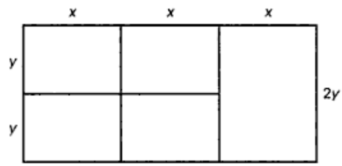
5. Find all the angles between and inclusive of 0° and 360° which satisfy the equation

   (i) $4 \sin y + \tan y = 0$,

   (ii) $5 \sec z + 3 \cos z = 16$. (8)

{T15: Differentiation}

6. A developer wishes to use 528 m of fencing to make five plots of land as shown in the diagram. Show that the total area of the land A m^2 is given by $A = 396x - 6x^2$. Given that x and y may vary, find the dimensions of each plot of land for which A is a maximum. (6)



{T16: Integration}

7. Given a curve for which $\frac{dy}{dx} = 15x^2 + kx$ where k is a constant and that it passes through (1, 3) and (0, 4), find

   (i) the equation of the curve,

   (ii) the x-coordinate of the stationary point on the curve. (6)

{T3: Quadratic Functions}

8. (a) Find the range of values of x for which $7x \leq 6 - 5x^2$.

   (b) Calculate the values of p for which the equation $6x^2 - 4px + 2p = 0$ has equal roots. (5)

{T5: Factors of Polynomials}

9. (a) Find the remainder when $2x^3 - 3x^2 - 72x - 18$ is divided by $x + 5$.

   (b) Given that $x + 1$ and $x - 2$ are factors of $2x^3 + ax^2 + bx - 2 = 0$, find the value of a and of b. Hence find the third factor of the expression. (10)

{T1: Sets}

10. In a cohort of 120 Secondary Three students who study both Mathematics and Additional Mathematics, it is found that

    88 students like Mathematics,

    79 students like Additional Mathematics,

    64 students like both Mathematics and Additional Mathematics,

    x students dislike both Mathematics and Additional Mathematics.

    (a) Draw a Venn diagram to illustrate this information.

    (b) Find the value of x. (4)

4. Graph of Inverse Function

The graphs of a function and its inverse are reflective of each other in the line $y = x$.

Example 4

Sketch, on the same diagram, the graphs of

(i) $h : x \mapsto x^2 + 1$ for the domain $0 \le x \le 2$;

(ii) h^{-1} for the domain $1 \le x \le 5$. (C)

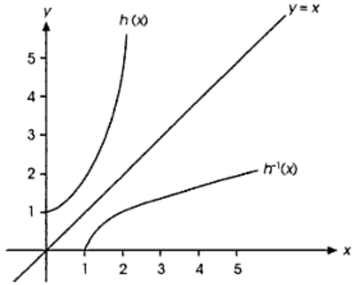
Solution

$h(0) = 0 + 1 = 1$
$h(1) = 1 + 1 = 2$
$h(2) = 4 + 1 = 5$

Let y be $h(x) = x^2 + 1$

$$x = \sqrt{y-1}$$
$$h^{-1}(x) = \sqrt{x-1}$$

$h^{-1}(1) = \sqrt{1-1} = 0$
$h^{-1}(5) = \sqrt{5-1} = 2$
$h^{-1}(2) = \sqrt{2-1} = 1$



5. Absolute Valued Functions

The **absolute valued function** is denoted by $|f(x)|$ where

$$|f(x)| = \begin{cases} f(x) \text{ if } f(x) \ge 0 \\ -f(x) \text{ if } f(x) < 0. \end{cases}$$

Any part of $f(x)$ that lies below the x axis is reflected in the line $y = 0$ (i.e. the x axis).

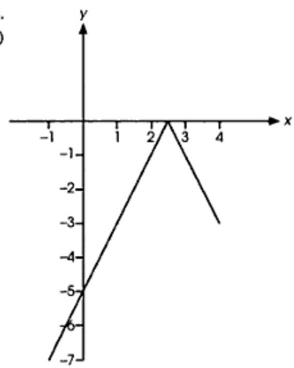
Example 5

Sketch the graph of $y = -|2x - 5|$ for $-1 \le x \le 4$. (C)

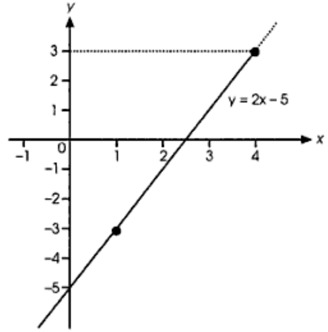
Solution

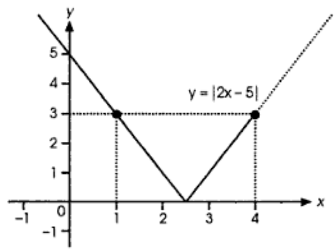
Let $f(x) = -|2x - 5|$
$f(-1) = -|2(-1)-5| = -7$
$f(0) = -|2(0)-5| = -5$
$f(4) = -|2(4)-5| = -3$

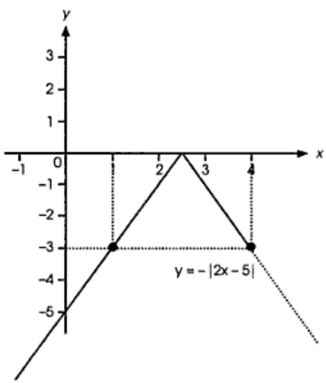
When $f(x) = 0$
$\Rightarrow 2x - 5 = 0$
$x = 5/2$



Alternatively, we can first sketch the straight-line graph of $y = 2x - 5$ (with y-intersect of -5, x-intersect of 2.5 and gradient of 2). Then, the graph is reflected about the x-axis to get the $y = |2x - 5|$ graph such that the portion of graph for $x < 2.5$ is 'bent upwards'. Next, to get $y = -|2x - 5|$, the $y = |2x - 5|$ graph is again reflected about the x-axis.

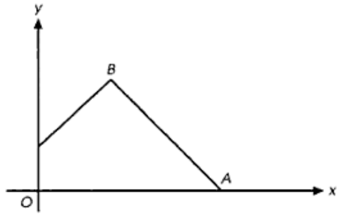






Example 6

The diagram shows part of the graph of $y = 3 - |x-2|$.



Find the coordinates of the points A and B. (C)

Solution At A, $y = 0$.

$\therefore\ 3 - |x-2| = 0$

$|x-2| = 3$

$x - 2 = \pm 3$

$x = 5$ or -1

Since it is shown in the diagram that A lies on the positive x axis, $x = 5$.

$\therefore$ The coordinates of A are $(5, 0)$.

At B, y is maximum.

When $|x-2| = 0$, i.e. $x = 2$,

$y = 3$ (maximum value of y).

$\therefore$ The coordinates of B are $(2, 3)$.

Example 7

(a) Functions f and g are defined by

$\mathrm{f}: x \mapsto \dfrac{6}{x-2},\ x \neq 2,$

$\mathrm{g}: x \mapsto kx^2 - 1$, where k is a constant.

(i) Given that $\mathrm{gf}(5) = 7$, evaluate k.

(ii) Express $\mathrm{f}^2(x)$ in the form $\dfrac{ax+b}{c-x}$, stating the values of a, b and c.

(b) On graph paper, using the same scale on each axis, draw the graph of $\mathrm{h}: x \mapsto \dfrac{2x+2}{x+2}$ for the domain $-1 \leq x \leq 3$.

(i) By drawing the appropriate straight line on the graph, obtain a solution of the equation $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$.

(ii) Using the same axes as for $\mathrm{h}(x)$, draw, on the same diagram, the graph of $\mathrm{h}^{-1}(x)$.

(iii) State the domain of $\mathrm{h}^{-1}(x)$. (C)

Solution (a) (i) $gf : x \mapsto g(f(x))$

$$= g\left(\frac{6}{x-2}\right)$$

$$= k\left(\frac{6}{x-2}\right)^2 - 1$$

$$= k\left(\frac{36}{x^2-4x+4}\right) - 1$$

$$gf(5) = k\left(\frac{36}{25-20+4}\right) - 1 = 7$$

$$k\left(\frac{36}{9}\right) = 8$$

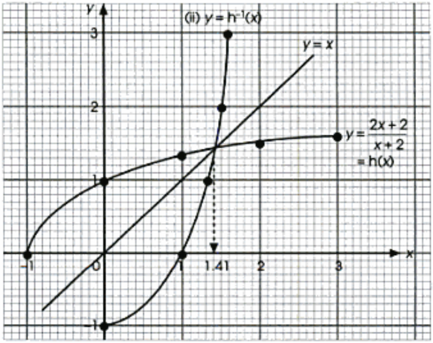
$$k = 2$$

(ii) $$f^2(x) = \frac{6}{\left(\frac{6}{x-2}\right)-2} = \frac{6}{\left(\frac{6-2x+4}{x-2}\right)} = \frac{6x-12}{10-2x} = \frac{3x-6}{5-x} = \frac{ax+b}{c-x}$$

$\therefore\ a = 3,\ b = -6$ and $c = 5$

(b) (i) Let y be $h : x \mapsto \dfrac{2x+2}{x+2}$.

x	-1	0	1	2	3
y	0	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$



When $h(x) = h^{-1}(x) \Rightarrow y = x$

The intersection of the curve $y = h(x)$ and the line $y = x$ gives the solution of $h(x) = h^{-1}(x)$.
From the diagram, $x = 1.4$ units

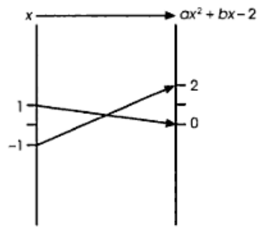
(ii) Let y' be $h^{-1}(x)$.

x	0	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$
y	−1	0	1	2	3

(iii) Domain of $h^{-1}(x)$ is $0 \le x \le \frac{8}{5}$

Revision Exercises

1.



The arrow diagram above shows part of the function $x \mapsto ax^2 + bx - 2$ where x is any real number. Calculate

(a) the value of a and of b,

(b) another element, apart from 1, that is mapped onto 0.

2. (a) State the minimum value of $2 - (x + 1)^2$ and state the corresponding value of x.

(b) Sketch the graph of the function $f : x \mapsto 2 - (x + 1)^2$ for the domain $-3 \le x < 2$ and write down the range of f corresponding to this domain.

3. For each of the following pairs of functions, express in similar form (i) fg, (ii) gf, (iii) ff and (iv) gg.

(a) $f : x \mapsto x + 1;\ x \ne -1;\ g : x \mapsto 2x^2 + 2$

(b) $f : x \mapsto x^2 - 1;\ g : x \mapsto \sqrt{x+1}$

(c) $f : x \mapsto \frac{3}{x} + 2,\ x \ne 0;\ g : x \mapsto 3 - 2x - x^2$

4. If $g : x \mapsto x + 2$ and $fg : x \mapsto -\frac{x}{x+2}$, find the function f.

5. The function f is defined by $f : x \mapsto 1 - \frac{1}{x}$ when $x \neq 0$. Find the functions f^2, f^3 and hence write down the functions f^{21} and f^{24}.

6. The function f is defined by $f : x \mapsto \frac{x^2}{2}$ for $0 \leq x < 4$. Sketch the graphs of f and f^{-1} on the same diagram. State the domain of f^{-1}.

7. The function f is defined as $f : x \mapsto \frac{ax+1}{x+2}$ and the $f(1) = 1$, find the value of a and of $f^{-1}(3)$.

8. Using graph paper, draw the graph of $y = |x^2 - x - 2|$ for the domain $-1 \leq x \leq 3$. Use your graph to estimate the solutions of the equation $|x^2 - x - 2| = x$.

9. (a) Find the range of the function $f : x \mapsto \frac{18}{x} + 8x$ for the domain $1 \leq x \leq 3$.
   (b) The function g is defined by $g : x \to 8 - 3x$. Find
      (i) an expression for $g^{-1}(x)$ and for $g^2(x)$,
      (i) the value of x for which $g^{-1}(x) = g^2(x)$.
   (c) The function h is defined by $h : x \mapsto ax + b$, $a \neq -1$. Given that the graph of $y = h(x)$ passes through the point (8, 5) and that the graphs of $y = h(x)$ and $y = h^{-1}(x)$ intersect at the point whose x- coordinate is 3, find the value of a and of b. (C)

10. Using graph paper, draw accurately on the same diagram, for $-3 \leq x \leq 3$, the graphs of $y = |x+2|$ and $2y = |x-4|$. Hence or otherwise, solve the equation $\frac{|x-4|}{2} = |x+2|$.

Chapter 1

Sets

Curriculum Objectives:

- Use set language and notation, and Venn diagrams to describe sets and represent relationships between sets as follows:
  $A = \{x : x \text{ is a natural number}\}$
  $B = \{(x, y) : y = mx + c\}$
  $C = \{x : a \leq x \leq b\}$
  $D = \{a, b, c, \ldots\}$
- Understand and use the following notation:

Union of A and B	$A \cup B$
Intersection of A and B	$A \cap B$
Number of elements in set A	$n(A)$
"... is an element of ..."	$\in$
"... is not an element of ..."	$\notin$
Complement of set A	A'
The empty set	$\varnothing$
Universal set	ε
A is a subset of B	$A \subseteq B$
A is a proper subset of B	$A \subset B$
A is a not a subset of B	$A \nsubseteq B$
A is a not a proper subset of B	$A \not\subset B$

A collection of objects is called a **set**. Each of the objects is an **element** of the set.

For example, the alphabet is a set and all the 26 letters are elements of the set. Denoting the alphabet by A, the set is represented as

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

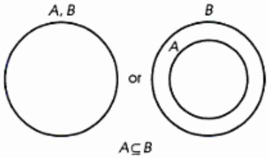
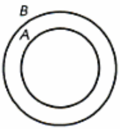
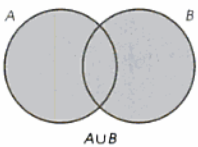
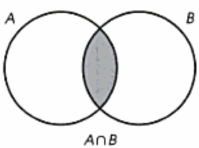
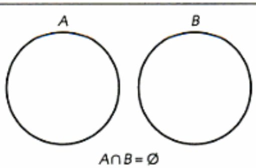
A statement like the letter "q is an element of A" can be written as $q \in A$ where $\in$ means 'is an element of'.

The number 3 is not an element of A and this can be written as $3 \notin A$ where $\notin$ means 'is not an element of'.

Examples of some widely-used mathematical set notations are:

$\mathbb{N}$ is the set of positive integers and zero, i.e.
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z}$ is the set of integers, i.e. $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
$\mathbb{Z}^+$ is the set of positive integers, i.e. $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

Set notation	Meaning	Venn diagram
$A = B$	Set A is **equal** to set B when they have the same elements.	
$A \subseteq B$	Set A is a **subset** of set B when every element in A is found in B.	
$A \subset B$	Set A is a **proper subset** of set B when every element in A is found in B and $A \neq B$.	
$A \cup B$	The **union** of sets A and B consists of all element found in A or B.	
$A \cap B$	The **intersection** of sets A and B consists of common elements found in both sets.	
$A \cap B = \varnothing$	Set A and set B are **disjoint** sets when no elements are common to both sets.	

Other set notations used are:

$\varnothing$ which means 'an empty set'; it has not elements;
n(A) which means 'the number of elements in set A';
$\nsubseteq$ which means 'is not a subset of';
$\not\subset$ which means 'is not a proper subset of'.

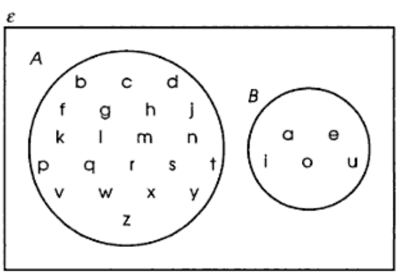
Example 1

All the letters of the alphabet are elements of the universal set. Set A consists of all the consonants and set B consists of all the vowels.

(a) List all the elements of set A and set B.
(b) Draw a Venn diagram.
(c) State whether the following statements are true or false.
 (i) $A = B$
 (ii) $A = B'$
 (iii) $A \subseteq B$
 (iv) $A \subsetneq \varepsilon$
 (v) $B \subset \varepsilon$
 (vi) $A = \varepsilon$
 (vii) $A \cap B = \varnothing$
 (viii) $A \cup B = \varepsilon$

Solution (a) A = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}
B = {a, e, i, o, u}

(b)



(c)

(i)	$A = B$	false
(ii)	$A = B'$	true
(iii)	$A \subseteq B$	false
(iv)	$A \subseteq \varepsilon$	true
(v)	$B \subset \varepsilon$	true
(vi)	$A = \varepsilon$	false
(vii)	$A \cap B = \varnothing$	true
(viii)	$A \cup B = \varepsilon$	true

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